THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY

R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

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WITH THE ASSISTANCE OF

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THE GROWTH OF MATHEMATICS

PRESIDENTIAL ADDRESS TO THE MATHEMATICAL ASSOCIATION, JANUARY 1957

BY G. TEMPLE.

MATHEMATICS is commonly regarded as a subject in which deductive reasoning reigns supreme, but the history of the growth of mathematics is far from being a record of systematic logical development. In fact the main features of the history of mathematics are the impact of the natural sciences, the transmission of the mathematical tradition, and its enrichment by the imaginative and intuitive efforts of individual mathematicians. The purpose of this lecture is to exemplify this statement by selective illustration from different branches of pure and applied mathematics, and to examine some of its practical consequences in the art of teaching.

Mathematics is an abstract study, and the history of mathematics is a history of abstraction. But the nature of mathematical abstraction is frequently misunderstood, and hence flow many misconceptions and difficulties for the student. To say that mathematics is abstract does not mean that it is remote and isolated from the real world of concrete individual things. The mathematical world is not a different world from the physical universe. It is the same world but viewed in a different light—an intellectual light which reveals much that is hidden in the light of common day.

That multivalent word, " abstraction ", so rich in its psychological and philosophical associations, still remains charged with an earthy flavour. It suggests a mental activity which extracts secrets from Nature like a man drawing buckets of water from a well, or which purifies our concepts like a chemist distilling a quintessence. Mathematical abstraction has its own special characteristics, but one feature it does share with philosophical abstraction. This vitally important feature is that the concepts obtained by mathematical abstraction are apprehended only when considered with reference to the material from which they are abstracted. Abstraction separates but does not isolate.

It is therefore appropriate to begin our survey of the intrinsic pattern of the growth of mathematics by a consideration of the kind of material available at any epoch to the mathematician engaged in active research. And we

shall single out for especial mention two kinds of material—the existing traditions of the craft and the challenge presented by the natural sciences. In the second place we shall consider how this material is utilised by two species of abstraction of particular significance in mathematics, which I will call abstraction by exclusion and abstraction by extension. The study of mathematics in the school or university is a kind of recapitulation of the past history of mathematics. The elucidation of the principles operative in the historical development should therefore have a significance, either immediate or remote, for the teacher and lecturer in mathematics.

Let us then turn first to the material factors which I have mentioned—mathematical tradition and physical problems. It is especially appropriate that here we should begin with what must be the primary influence on the development of mathematics—the records of past achievements as enshrined in books and expounded by teachers. It is well known how the development of analysis in this country has been decisively influenced by G. H. Hardy's epoch-making textbook entitled *A Course of Pure Mathematics*. And it seems probable that our whole approach to algebra, analysis and topology will be transformed by the astonishing series of monographs by that group of French mathematicians who write under the collective synonym of " Bourbaki ". It would not be difficult to find counter-examples of branches of mathematics which have in the past become almost moribund for the lack of good text-books. But I will say no more than to contrast the puerile trivialities of some mathematical textbooks on statics with the vital and inspiring engineering texts on the theory of structures.

The great classics of mathematics are naturally beyond the capacity of the schoolboy, but even his elementary lessons can be enriched and enlivened by appropriate references to the history of the subject ; and teachers of mathematics are becoming increasingly aware of the inspiration which they can derive from some knowledge of the historical development of mathematical concepts and methods. One of the lamentable features of contemporary university life is the decline in the discipline of the study of the mathematical classics, and the substitution of ephemeral lecture notes for the abiding companionship of a personally-acquired library, however small it may be.

It is not only by their writings that the great mathematicians have influenced the development of the subject, but also in their personal influence on their students and colleagues. I need mention only the famous school of analysis established in Cambridge by Hardy and Littlewood, and the great number of mathematical physicists who owe their training and inspiration to Sir Geoffrey Taylor.

To mention the name of this great mathematician, meteorologist, physicist and engineer brings me to the second material influence on the growth of mathematics—the challenge of the problems presented by the world of nature and art.

It is well known how the contemplation of the geometry of a tiled floor led Pythagoras to discover his famous theorem on the right-angled triangle, and how the study of the vibrating violin string led to the theory of Fourier series. Every branch of applied mathematics has sprung from the endeavour to express in an organised philosophy the complex events of the real external world—although there are certain textbooks which would lead one to believe that applied mathematics is a department of logic! During the last few years a striking example of the growth of a purely mathematical discipline, stimulated—in fact provoked—by the demands of the theoretical physicist, is the invention of the Theory of Distributions, which has given a simple, coherent and consistent expression to the vague, self-contradictory but eminently useful " improper functions " and illicit inferences of quantum theory and wave theory.

The practical requirements of the engineer and his insistent demands for methods of rapid numerical calculations have led to the invention of new techniques for the solution of mathematical problems, such as the method of relaxation, and have sometimes given new significance to the abstract existence theorems of the pure mathematician.

Modern methods of teaching mathematics are well aware of the importance of developing the power of abstraction from concrete material objects and events. Geometry is no longer imposed as a rigid dogmatic system, but is developed by the co-operation of student and teacher in progressive inferences, inductions and abstractions from the study of simple geometrical patterns. The student of applied mathematics is to be found in the laboratory, and the distinction between applied mathematics and theoretical physics is hard to maintain.

From this cursory sketch of some of the external factors which influence the growth of mathematics we must pass to a rapid review of some of the internal mental processes by which the individual mathematician advances his subject. I shall refer in particular to two kinds of abstraction which are of special significance in the development of mathematics—abstraction by exclusion and abstraction by extension, and first I will speak of abstraction by exclusion.

Confronted by the bewildering complexity of the real world the scientist begins by restricting his studies to those objects which can be known, at least in part, without explicit reference to the rest of the universe. Thus, on different scales of operation, we speak of the solar system, the galactic system, the system of extra galactic nebulae ; we have theories of the structure of a single atom, a molecule, or of radiation in an isolating enclosure. The concept of a " system " implies that we are prescinding from the effect of all external influences. Such an intellectual exercise may yield results which are fruitful or illusory. Only subsequent experience can be our guide and critic. The effect of the rotation of the Earth can be ignored on a firing range or in aerodynamics, but not in the study of Foucault's pendulum or the gyro compass.

This simplification of an object of study by the neglect of its environment is an obvious example of what I have called abstraction by exclusion. In mathematical studies this method is still further developed by the systematic suppression of certain features of an object of study in the mental pictures which we form. This process is commonly called " idealisation ", and its methodical use is a notable characteristic of applied mathematics.

An outstanding example of the success of idealisation, or abstraction, by exclusion, is provided by the modern development of aerodynamics. The subject of study in aerodynamics is indeed a fluid, but a mathematical fluid and not the air in which we move, breathe and fly. The idealised mathematical fluid has been deprived by abstraction of its compressibility, of its viscosity, of its thermal conductivity, of its molecular structure—indeed of every physical property save its density, its fluidity and capacity for exerting pressure. And yet this attenuated ghost of the real air provides a most suitable and appropriate mental model. Its properties are much more readily investigated than those of the physical air with its complex structure and characteristics. And, in a wide range of conditions, the types of motion produced in the perfect mathematical fluid closely simulate the motion of the real air. The success of the abstract theory of the perfect fluid is as complete as it is unmerited.

The explanation of the success of this idealised fluid dynamics is one of the triumphs of the German aerodynamicist, Prandtl—and I refer to it now because it provides another example of successful idealisation. Prandtl showed that, in the motion of a solid body such as an arrow, a missile, or an

aircraft, through the real atmosphere, the physical effects of viscosity and thermal conductivity are confined to a thin boundary layer on the surface of the moving body, and to the wake behind the body. Elsewhere the behaviour of the real air conforms to that of the perfect fluid. In the real air there is no sharp transition from the regime of a boundary layer to the regime outside, but in the current idealisation the concept of a boundary layer with a definite thickness forms a most valuable instrument of thought.

In the realm of pedagogics the doctrine of abstraction by exclusion provides a philosophic justification for the kind of examples and exercises which mathematics students in this country are encouraged to attempt. (I say advisedly " in this country " for the Continental tradition in this matter was quite different.) The principles of dynamics are few and simple, but it is impossible for a student to appreciate their power until he has applied them in numerous problems. But the real problems of physics and engineering are usually far too complex and difficult for this purpose. And therefore idealised problems must be devised in order to provide a suitable training for the novice. This is the justification of the artificialities in our textbooks which arouse the amusement, contempt or disgust of the casual reader. Surfaces which may be " perfectly smooth " or " perfectly rough ", bodies of " negligible mass ", " inextensible and perfectly flexible " strings—all these are idealisations which serve a most useful purpose in exercising the powers of the student.

Any criticism of the use of these abstractions, precisely because they are abstractions, is quite mistaken. But there are two real criticisms which can be made of these traditional idealised examples. The first is that the student is never, or most seldom, encouraged to make the idealisations himself. It would be of the greatest value to the best students if they were called upon to construct their own idealised versions of practical physical problems. Perhaps the dynamical problems which arise in sport and athletics would provide suitable material for this purpose. The second criticism is that the student is almost invariably told what he is to prove, whereas in real research it is the formulation of the right question which is difficult as compared with its subsequent solution. From time to time exercises might be set in the most general terms so as to stimulate the faculties of abstraction and curiosity. For example, " Discuss the spinning of a penny," " Discuss the rise of bubbles in water," " Discuss the vibrations of railway carriages."

So far our illustrations of abstraction by exclusion have been drawn from applied mathematics, but there is an analogous mental process of proved value in pure mathematics, I mean the abstraction by axiomatisation. There comes a stage in the development of a new subject when our intuition fails, our logical powers seem ineffective and our resources are exhausted. This is the time for a radical stocktaking of our equipment by an explicit cataloguing of all our undefined notions and unproved assumptions and a systematic and logical organisation of all the propositions which we have been able to deduce. This is the process of axiomatisation. In this process it is often advantageous to consider the terms and symbols in which the axioms are expressed as mere counters in a game, and to divest them temporarily of all significance except such as is conveyed by the axioms themselves. The mind of a mathematician seems sometimes to acquire new strength from such a diversion and distraction, and to perceive hitherto unnoticed paths of advance.

Here are two examples drawn from the theory of probability and from topology. The basic concepts and methods of the theory of probability are well known in an implicit and confused manner to all scientists, together with some acquaintance of the outstanding paradoxes of this theory, which naturally create suspicion of its value. One characteristic of the developments in statistics and probability during the last thirty years has been the explicit use of axiomatisation as an instrument of self-criticism and research. Per-

haps the most obvious and useful achievement of this method has been the replacement of the old and ill-defined concept of "probability" by the perfectly explicit concept of "relative probability".

To illustrate the need for axiomatisation in topology consider the simple statement that a plane closed curve, which does not intersect itself, has an inside and an outside. As soon as we begin the process of axiomatisation we realise the need for a definition of the term "curve"—and we soon discover that with some definitions there are curves which can fill an area. In order to provide a rigorous proof of the theorem just enunciated it is necessary to discard all our intuitive knowledge of curves and to start afresh with open minds to construct an axiomatic basis for topology.

I need scarcely say that axiomatisation is primarily an instrument of study and research for the advanced student and that it is quite out of place in elementary work, where its abstract character inevitably conduces to boredom and repulsion. The ancient methods of teaching geometry began by learning by rote the Euclidean axioms and postulates, for which the student was quite unable to see the need or significance. Happily modern methods postpone the study of the axioms of geometry to a later stage when the student begins to feel their necessity in order to coordinate his knowledge.

I now approach the most difficult part of my lecture and attempt to give some account of what is beyond doubt the distinctive mental operation by which mathematics is advanced—I mean abstraction by extension, invention or creation. The psychology of invention in the mathematical field has formed the title of an interesting essay by Hadamard (Princeton, 1949), and Henri Poincaré has given us a number of personal experiences which form precious material for the psychologist. No better introduction to the subject of abstraction by extension could be found than direct quotation describing the famous lecture which Poincaré gave to the Société de Psychologie in Paris on "Mathematical Creation".

"Poincaré's example is taken from one of his greatest discoveries, the first which has consecrated his glory, the theory of fuchsian groups and fuchsian functions. 'In the first place' says Hadamard, 'I must take Poincaré's own precaution and state that we shall have to use technical terms without the reader's needing to understand them. I shall say, for example,' he says, 'that I have found the demonstration of such a theorem under such circumstances. This theorem will have a barbarous name, unfamiliar to many, but that is unimportant; what is of interest for the psychologist is not the theorem, but the circumstances.'

"So, we are going to speak of fuchsian functions. At first, Poincaré attacked the subject vainly for a fortnight, attempting to prove that there could not be any such functions: an idea which was going to prove to be a false one.

"Indeed, during a night of sleeplessness and under conditions to which we shall come back, he builds up one first class of those functions. Then he wishes to find an expression for them. Poincaré tells us:

"'I wanted to represent these functions by the quotient of two series; this idea was perfectly conscious and deliberate; the analogy with elliptic functions guided me. I asked myself what properties these series must have if they existed, and succeeded without difficulty in forming the series I have called thetafuchsian.

"'Just at this time, I left Caen, where I was living, to go on a geologic excursion under the auspices of the School of Mines. The incidents of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I

had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the result at my leisure.

" 'Then I turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the seaside and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformations of indefinite ternary quadratic forms were identical with those of non-Euclidean geometry.' "

I should like to set side by side with Poincaré's lecture an extract from a celebrated letter by Mozart the musician in which he describes the suddenness and spontaneity of creative work in music:

"When I feel well and in a good humour, or when I am taking a drive or walking after a good meal, or in the night when I cannot sleep, thoughts crowd into my mind as easily as you could wish. Whence and how do they come? I do not know and I have nothing to do with it. Those which please me, I keep in my head and hum them; at least others have told me that I do so. Once I have my theme, another melody comes, linking itself to the first one, in accordance with the needs of the composition as a whole: the counterpoint, the part of each instrument, and all these melodic fragments at last produce the entire work. Then my soul is on fire with inspiration, if however nothing occurs to distract my attention. The work grows; I keep expanding it, conceiving it more and more clearly until I have the entire composition finished in my head though it may be long. Then my mind seizes it as a glance of my eye a beautiful picture or a handsome youth. It does not come to me successively, with its various parts worked out in detail, as they will be later on, but it is in its entirety that my imagination lets me hear it."

This lecture on the growth of mathematics would, I feel, have been incomplete without this citation of personal experience of a great mathematician. However, I shall say no more about the *nature* of the internal mental processes which lead to the higher forms of mathematical abstraction, but turn to a description of their external manifestations.

It is almost true to say that the history of mathematics is the record of successive generalisations. For example, we have the successive elaboration of the concepts of the signless integers, the positive and negative integers, rational fractions, real numbers, complex numbers. Each of these concepts is a generalisation of the preceding. This does not mean that each of these concepts is a particular subspecies of the succeeding concept. That would be a complete misunderstanding of the nature of abstraction by extension. The relation generated by extension is more subtle and more flexible, and it is worth while to attempt a brief description in order to bring out the creative character of mathematical work.

Consider the relation between the integers and fractions. What is a fraction? It is an ordered pair of integers (p, q) called respectively the numerator, p, and the denominator, q. It is therefore manifest that no fraction can be an integer. A fraction is a different species of mathematical entity. Nevertheless there is a certain class of fractions which behave like integers, to wit, fractions of the form $(p, 1)$ when the denominator is unity. Let us call these integral fractions. Then it is clear that there is a one-to-one correspondence between integers and integral fractions, and between any operations executed on integers and the analogous operation executed on

integral fractions. In the terminology of modern algebra, this is expressed by saying that the class of integers and the class of integral fractions are isomorphic, and that the class of fractions is an *extension* of the class of integers.

This simple example shows up all the characteristics of abstraction by extension. Firstly, the relation between a concept and its extension is susceptible of strict logical analysis and definition, *after* the extension has been invented by the mathematician. But until the generalisation or extension has been formulated its relation to the primitive concept is unknown, and the way in which the generalisation is gradually given expression can only be compared with the analogous processes of poetic experience.

Secondly, the relation of abstractive extension is certainly not elementary in character, and quite unsuitable for elementary teaching. The problem confronting the teacher is to elicit in the pupil the act of extensive abstraction —without any analysis of the nature of this act. He must teach fractions without any reference to the formal description of fractions which we have sketched above. This is a formidable task which must excite the admiration of those who are occupied with the far easier work of lecturing on differential equations or analytical dynamics.

Thirdly, it is worth noting that the invention of fractions enables us to answer questions which were unanswerable in the restricted language of integers. And this is typical of extensive abstraction in the history of mathematics. The pattern of mathematical progress frequently exhibits the two stages of frustration and invention—frustration at the discovery that certain problems are insoluble with existing concepts and methods, and then the invention of new and more general concepts, new and more powerful methods to resolve these problems. The whole theory of limiting processes is an excellent example of this pattern of abstraction.

Fourthly, there is a feature of the mental processes of abstraction which is of special interest for the teacher and lecturer—*i.e.* their avowedly tentative and experimental character in their initial stages. Such phrases as " imaginary numbers ", " improper functions ", " symbolic elements ", are relics of past periods of mathematical history when the processes of the creation or invention of complex numbers, distribution functions and nilpotent Aronhold Clebsch algebras were still at a stage of free experimentation unshackled by the factors of a formalised system.

The last characteristic of abstraction by creation which I will mention is the experience of wordless and imageless thought. It is certainly the testimony of many mathematicians, which I can confirm from my own experiences that the great problem is to find words, concepts and images which can embody the free thoughts which we are struggling to express. Perhaps a quotation from Francis Galton, the great geneticist, will elucidate this mysterious activity.

" It is a serious drawback to me in writing, and still more in explaining myself, that I do not so easily think in words as otherwise. It often happens that after being hard at work, and having arrived at results that are perfectly clear and satisfactory to myself, when I try to express them in language I feel that I must begin by putting myself upon quite another intellectual plane. I have to translate my thoughts into a language that does not run very evenly with them. I therefore waste a vast deal of time in seeking for appropriate words and phrases, and am conscious, when required to speak on a sudden, of being often very obscure through mere verbal maladroitness, and not through want of clearness of perception. That is one of the small annoyances of my life."

It is now time for me to draw to an end. I have been acutely conscious, during the preparation and delivery of this lecture, of the highly selective

nature of the material which I have presented; but I have endeavoured to describe some of the main factors which influence the growth of mathematics, and where possible to refer to their significance in the teaching of this subject. The treatment has been far from exhaustive and many themes have been left untouched. There is in particular one topic, which I have so far neglected, but to which I must refer by way of conclusion—the influence of aesthetic judgements not only in the final presentation of a new theory, but also in the inception and formation of the theory in our minds. The formal elegance of a mathematical proof is not an adventitious decoration superadded to a gaunt logical exposition. It is of the very essence of the work; and a sense of beauty, of the innate fitness of things, is an indispensable guide in that work of choice and selection which forms such a large part of mathematical activity. Moreover, aesthetic considerations can determine the very choice of the problem which a mathematician selects for study. Perhaps this is the clue to those difficult questions, how is mathematical abstraction to be acquired, how is mathematical technique to be learnt? Perhaps also it is the clue to a far more searching question—What is mathematics?

G. T.

MATHEMATICAL PROBLEMS OF ASTRONAUTICS *

By D. F. Lawden

1. *Introduction.* Astronautics is the scientific study of the many problems which arise when we contemplate the possibility of releasing man from the Earth's gravitational field, setting him free to explore the Solar System and later the regions beyond. Not so long ago, to admit having such ambitions was regarded as a symptom of megalomania. Mathematical " proofs " of the impossibility of escape from the Earth's field were legion and ranged from the simple statement " a rocket will not develop thrust in a vacuum " to a (perfectly correct) numerical calculation showing that the most powerful fuel possesses insufficient energy to release even its own mass from the Earth's attraction. It was made very clear to the early enthusiasts that few advances in this field could be made until the climate of scientific opinion changed and astronautics became " respectable ". It is, of course, quite normal for a line of scientific enquiry to evolve from the stage at which its ideas are the subject of ridicule to the stage at which they are universally acclaimed. The idea of the periodic classification of the elements caused our great-grandfathers to laugh hugely into their beards. Marconi was a notorious striver after the unattainable. The Wright brothers achieved the " impossible " and, in defiance of contemporary opinion, the aeroplane proceeded to change our way of life irrevocably. Astronautics has been singularly fortunate. After spending comparatively few years in the wilderness, the advent of the V.2 and other rocket weapons brought about that change in the attitude of scientists towards the subject which had been seen as a necessary preliminary to the serious consideration of the problems involved. Greatly decimated, the doubters now depend upon the slogan—" *it can be done, but it is not worth doing* ". However, it is not the purpose of this article to discuss the value, in human terms, of conquering space but only to explain briefly why and how it can be done if man so desires. In any case, it is unlikely that the real significance for mankind of such a tremendous advance can be appreciated before the event. Only the passing of the years can reveal to us the value of our actions.

2. *The Problems.* Although the possibility of space travel is not now seriously questioned, it must be admitted that the task is so complex that very

* This article is based upon a talk given to the Sheffield and Midland Branches of the Mathematical Association.

nearly all the branches of science, engineering and medicine must make a contribution before success can be assured. Mathematical techniques will, of course, be employed by all these sciences in the solution of their particular problems, but mathematics has its own distinctive contribution to make in the realm of space navigation. In the seventeenth and eighteenth centuries the problems of sea navigation offered a challenge to the mathematicians of those days, causing many productive researches to be undertaken and we may expect that the more complex problems associated with the computation of space trajectories will also have a stimulating effect upon the body of mathematical research.

There are three main problems of space navigation. Firstly, there is the problem of computing the most convenient trajectory to be followed by a space ship undertaking a particular interplanetary journey. This trajectory will clearly be that requiring the least expenditure of fuel after various limiting conditions have been satisfied. Typical limiting factors are the maximum motor thrust available, the minimum distance of approach to the Sun and the maximum allowable time for the journey. In the early stages these conditions will have to be relaxed to the maximum degree possible in order to achieve fuel economy, but when atomic drives become a reality the calculation of trajectories of least fuel expenditure will have to be carried out under a variety of conditions imposed by considerations of time schedules and passenger safety.

The second main problem is that of computing the position and velocity of a rocket in space from observations of celestial bodies and the subsequent derivation of the elements of its orbit. This is a fairly straightforward survey resection problem and the existing apparatus of spherical trigonometry will be quite capable of providing a solution. The directions of the various bodies of the Solar System can be found by observing them against the background of the stars and then, knowing the coordinates of these bodies, those of the rocket are easily derived.

Thirdly, there is the problem of providing the space navigator with sets of tables, by the aid of which he will be able to compute the appropriate manoeuvre necessary to bring his ship back on to a pre-calculated track when a divergence has been observed.

The first of these problems is of most mathematical interest and it is the one with which we shall be chiefly concerned.

3. *Elementary Considerations.* It is well known by now that the only available device which can develop an appreciable thrust in free space is the rocket motor. This comprises a combustion chamber into which a liquid fuel and an oxidant are pumped and where they combine with the liberation of heat and consequent rapid expansion of the products of combustion. The V.2 rocket employed alcohol as a fuel and liquid oxygen as an oxidant. Other propellant combinations which have been used are, aniline and nitric acid (the American Aerobee), and methanol and liquid oxygen (the Armstrong Siddeley " Snarler "). The most powerful combination known comprises liquid hydrogen and liquid fluorine, but their product of combustion is so highly corrosive

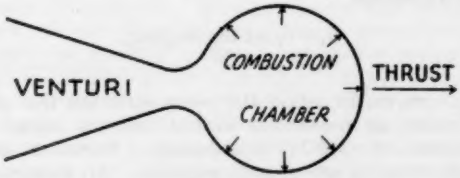

Fig 1.

This article is based upon a talk given to the Sheffield and Midland Branches of the Mathematical Association.

that there is little likelihood of any early practical application. The expanding gases leave the motor via the venturi (Fig. 1). If the venturi were closed, the pressure distribution over the walls of the combustion chamber would have zero resultant and no thrust would be observed. By opening the venturi, the pressure across the exit is reduced and a resultant thrust commences to operate in the opposite direction.

Mathematically, let m be the mass of the rocket at time t and $\mathbf{v}$ its velocity. Let $\mathbf{c}$ be the exhaust velocity. Let $\mathbf{F}$ be the resultant force on the rocket due

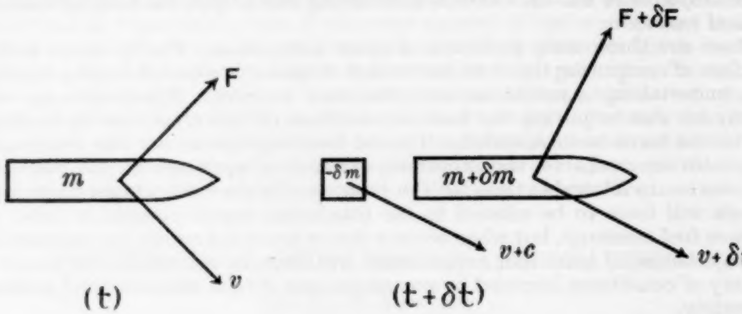

Fig. 2.

to all agencies but its own motor (*e.g.* gravity and atmospheric resistance.) During the time interval $(t, t+\delta t)$ the situation develops as shown in Fig. 2. The impulse equation for the system of particles comprising the rocket at time t over the period of time $(t, t+\delta t)$ is

$$\mathbf{F}\,\delta t = (m+\delta m)(\mathbf{v}+\delta\mathbf{v}) - \delta m(\mathbf{v}+\mathbf{c}) - m\mathbf{v},$$
$$= m\,\delta\mathbf{v} - \mathbf{c}\,\delta m,$$

having neglected quantities of the second order. Dividing throughout by δt and proceeding to the limit, we obtain the equation of motion

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{c}\frac{dm}{dt}, \qquad (1)$$

where $\mathbf{c}\,dm/dt$ is the motor thrust.

Consider a rocket in rectilinear motion in empty space where $\mathbf{F}=0$. Since the vectors $\mathbf{v}$ and $\mathbf{c}$ are now in opposite senses, equation (1) is equivalent to the scalar equation

$$m\frac{dv}{dt} = -c\frac{dm}{dt} \quad\text{or}\quad dv = -\frac{c\,dm}{m}. \qquad (2)$$

Suppose the motor is run for some time during which the velocity increases from v_0 to v_1 and the mass decreases from m_0 to m_1. Integrating equation (2) over this interval we obtain

$$v_1 - v_0 = c\log_e(m_0/m_1),$$
or
$$m_0/m_1 = e^{\Delta v/c}, \qquad (3)$$

where $\Delta v = v_1 - v_0$. m_0/m_1 is called the *mass ratio* for the manoeuvre. We observe that to cause an increment in the velocity equal to the exhaust velocity c, a mass ratio of $e = 2{\cdot}718$ is necessary. However, increments in excess of the exhaust velocity are clearly possible. An increment of $2c$ can be attained with a mass ratio of $e^2 = 7{\cdot}39$, and so on. The mass ratio of the V.2

Fig 1.

corresponding to a complete expenditure of fuel is about 3. The dead structural weight cannot be reduced below a certain minimum without danger of failure under the stresses arising from the motor thrust and engineers do not therefore regard mass ratios in excess of 6 as practical propositions. The maximum velocity increment which can be obtained from a rocket is therefore about log $6(=1\cdot 8)$ times its exhaust velocity.

Neglecting atmospheric resistance, a body given a velocity of 11·2 km/sec at the Earth's surface would recede to infinity along a parabolic arc. This is the *escape velocity*. Unless we can harness atomic energy to drive a rocket motor, exhaust velocities of more than 4 km/sec are unlikely to be attained. The maximum velocity increment to be expected from a rocket is accordingly about $4 \times 1\cdot 8 = 7\cdot 2$ km/sec. This is insufficient to achieve escape. In addition, it must be remembered that we have made no allowance for velocity losses due to air resistance and gravity. A pessimistic conclusion as to the imminence of the conquest of space would therefore appear to be justified and those who wished to be so persuaded did not ignore the opportunity offered by the facts we have here presented. However, there are two methods available for resolving the difficulty and both were well understood at least two decades ago.

If we design a rocket which carries as its payload a second rocket and we start the motor of the second rocket functioning at the instant when the motor of the first rocket ceases to fire, assuming that each rocket has maximum mass ratio and maximum exhaust velocity, the second rocket will experience a velocity increment of $2 \times 7\cdot 2 = 14\cdot 4$ km/sec. This is sufficient for escape. Employing a two-step rocket of this type (V.2 + WAC-Corporal) the Americans achieved a height of 250 miles in 1949. Engineers are prepared to consider the possibility of three-step rockets. This makes space travel a feasible proposition, even today.

The second method of easing the situation is to limit the first objective to the placing of a rocket in a circular orbit about the Earth. The velocity in such an orbit just beyond the Earth's atmosphere is $1/\sqrt{2}$ times the escape velocity, *viz.* $11\cdot 2/\sqrt{2} = 7\cdot 92$ km/sec. This velocity is only just beyond the capability of an ideal single stage rocket and a three-step combination will shortly be employed by both the Americans and the Russians to place a payload of a few pounds of instruments into such a circular orbit. As a first step towards the ultimate conquest of space, it is therefore suggested that, over a period of some years, a number of rockets should be placed in a circular orbit, carrying as cargo parts from which a space ship could be constructed and propellant with which to energise its motors. The ship would be of light construction, since it will be effectively weightless until its motors are activated and the thrust these must develop need only be relatively small. Each ferry rocket would be guided towards its rendezvous with those fired earlier either by radio control from the ground or by means of its own homing device. The final, and most difficult stage of the enterprise, would be to transport into the orbit men entrusted with the task of fabricating the ship and projecting it into space. Once constructed, a velocity increment of $11\cdot 2 - 7\cdot 9 = 3\cdot 3$ km/sec would release the ship from the Earth's field to follow a trajectory into space.

4. *Optimal Rocket Trajectories.* Having shown the feasibility of escape from the Earth, we will now discuss the problem of optimal rocket trajectories. This is the fundamental problem of space navigation and may be stated as follows : A rocket possesses a given velocity at a point A of a specified gravitational field. It is required to manoeuvre the rocket so that it arrives at a terminal point B with a specified velocity and minimum fuel expenditure. The instants of departure and arrival, may or may not be specified.

This problem appears, upon a superficial examination to be a typical problem in the calculus of variations. However, a difficulty arises which makes the

problem very much more interesting from the mathematical point of view. To illustrate this remark, we will consider the simple case of a rocket which is to be transferred from a point O to another A in the absence of air resistance and gravity. For further simplicity, we will suppose the motion to take place in a plane.

Let Ox, Oy be axes in the plane of the motion and let $(0, 0)$, (a, b) be the terminals (Fig. 3). It is required to manoeuvre a rocket between these points

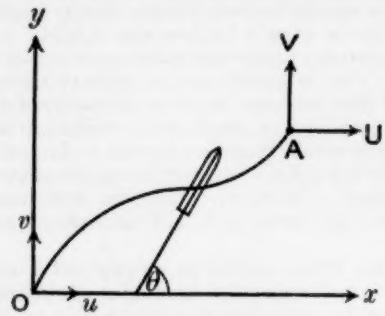

FIG. 3.

so that, leaving the terminal O with velocity (u, v) it arrives at A with velocity (U, V), the fuel expenditure to be a minimum.

Suppose the rocket leaves O at $t=0$ and arrives at A at $t=T$. We shall regard the time of transit as a given quantity, as it very often will be in practice. If T is open to choice, we can minimise later with respect to this parameter.

Resolving equation (1) parallel to the axes Ox, Oy, we obtain the equations of motion

$$m\ddot{x} = -c\cos\theta\frac{dm}{dt},\quad m\ddot{y} = -c\sin\theta\frac{dm}{dt}, \quad \text{.....................(4)}$$

where θ is the angle made by the direction of the motor thrust with the x-axis. From these equations, eliminating θ, we find that

$$\frac{c}{m}\frac{dm}{dt} = -\sqrt{\ddot{x}^2+\ddot{y}^2}. \quad \text{...............................(5)}$$

Integrating over the time interval $(0, T)$ we obtain

$$c\log\frac{m_0}{m_A} = \int_0^T \sqrt{\ddot{x}^2+\ddot{y}^2}\,dt \quad \text{...............................(6)}$$

and the fuel expenditure is a minimum if m_0/m_A is a minimum, *i.e.* if the integral is a minimum. We have therefore to choose the functions $x(t)$, $y(t)$, determining the trajectory, in such a way that the above integral is minimised. The end conditions on these functions are

$$\left.\begin{array}{ll} x=y=0,\ \dot{x}=u,\ \dot{y}=v, & \text{at } t=0, \\ x=a,\ y=b,\ \dot{x}=U,\ \dot{y}=V, & \text{at } t=T. \end{array}\right\} \quad \text{........................(7)}$$

Applying Euler's method for the minimisation of an integral of this type

under the stated boundary conditions, we obtain the characteristic equations

$$\frac{d^2}{dt^2}\left\{\frac{\ddot{x}}{\sqrt{(\ddot{x}^2+\ddot{y}^2)}}\right\}=\frac{d^2}{dt^2}\left\{\frac{\ddot{y}}{\sqrt{(\ddot{x}^2+\ddot{y}^2)}}\right\}=0,$$

or

$$\frac{d^2}{dt^2}(\cos\theta)=\frac{d^2}{dt^2}(\sin\theta)=0. \qquad (8)$$

Hence

$$\cos\theta=At+B,\ \sin\theta=Ct+D, \qquad (9)$$

where A, B, C, D are arbitrary constants. Since $\cos^2\theta+\sin^2\theta=1$ identically, these equations can only be true if $A=C=0$ and $B^2+D^2=1$, *i.e.* if $\theta=$constant. Thus the result obtained is that the thrust must have an invariant direction. It is easy to show that, in general, a manoeuvre carried out under this condition cannot be successful. If the thrust is aligned so that the required velocity change takes place in time T, the displacement will, in general, be incorrect. If, on the other hand, the thrust is directed to yield the required displacement, the velocity change will be unsatisfactory. Euler's method does not, therefore, give us a solution in the general case.

The reason why the orthodox method fails, becomes clear so soon as the actual solution to the problem has been found. This may be discovered as follows :

Let $p=\dot{x}$, $q=\dot{y}$, so that

$$I=\int_0^T\sqrt{\dot{p}^2+\dot{q}^2}\,dt \qquad (10)$$

has to be minimised by choice of the functions $p(t)$, $q(t)$ subject to the conditions

$$\begin{aligned}p=u,\ q=v\quad &\text{at } t=0,\\ p=U,\ q=V\quad &\text{at } t=T,\end{aligned}$$

$$\int_0^T p\,dt=a,\quad \int_0^T q\,dt=b. \qquad (11)$$

The functions $p(t)$, $q(t)$ determine a curve in the hodograph plane (Fig. 4) which must connect the points $J(u, v)$, $K(U, V)$ and satisfy the pair of conditions (11).

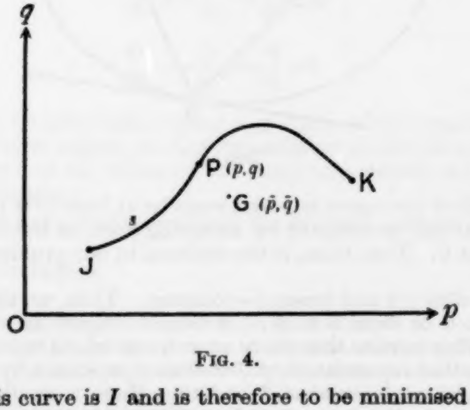

FIG. 4.

The length of this curve is I and is therefore to be minimised subject to conditions (11).

Consider a distribution of matter along this curve of line density $\rho = dt/ds$, where s is the arc length parameter measured from J. Since t increases with s, ρ is positive. Then the total mass of matter is

$$\int_0^I \rho\, ds = \int_0^T dt = T \quad \text{.................................(12)}$$

and is constant. Also

$$\int_0^T p\, dt = \int_0^I p \frac{dt}{ds} ds = \int_0^I \rho p\, ds = T\bar{p}, \quad \text{.....................(13)}$$

where $(\bar{p}, \bar{q})$ is the mass centre G. Similarly

$$\int_0^T q\, dt = T\bar{q}. \quad \text{....................................(14)}$$

Thus, the conditions (11) are equivalent to

$$\bar{p} = a/T, \quad \bar{q} = b/T, \quad \text{...................................(15)}$$

i.e. to fixing the position of G.

We have, therefore, to choose the curve of minimum length connecting two given points J, K along which it is possible to distribute matter of total mass T such that the mass centre is at a given point G.

Consider the ellipse with foci at J and K, passing through G (Fig. 5). Any curve joining J and K of length less than $(JG + GK)$ lies wholly inside the ellipse and has no part on the side of the tangent at G remote from JK. It follows that any distribution of matter along it cannot have its mass centre at

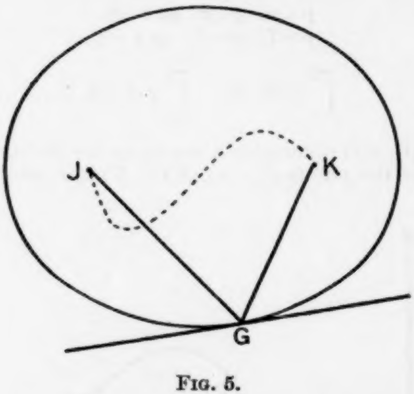

FIG. 5.

G and the length of the curve required must be at least $(JG + GK)$. However, this lower bound can be attained by accepting JGK as the curve and placing all the matter at G. This, then, is the solution to our problem and it remains to interpret it.

Where $\rho = 0$, $dt/ds = 0$ and hence t = constant. Thus, no time elapses as we move from J to G or from G to K. A time T elapses as we move over the particle at G. This implies that there must be an initial instantaneous change in velocity from that represented by J to that represented by G. The velocity now remains constant for a time T and then there is another instantaneous change from the velocity corresponding to G to that corresponding to K. The

interpretation of our solution is therefore that fuel must be consumed very rapidly when the rocket is at O, so that the rocket is subjected to an impulsive thrust changing its velocity into that necessary for it to be able to coast with uniform speed from O to A in the time T. During the motion from O to A, the motors are not energised, but upon arrival at A a second impulsive thrust converts the rocket velocity into that required at this point. It is clear that this manoeuvre involves discontinuities in the first derivatives of the rocket's coordinates and this explains why the orthodox method fails.

The solution to the more general problem of the transfer of a rocket between two points of any gravitational field, air resistance being neglected, can now be appreciated. If A and B are the two terminals at which the rocket velocity is specified, the optimal track joining these points comprises a number of null-thrust arcs along which the rocket falls freely under gravity, impulsive thrusts from the motor being applied at the junctions between these arcs to effect transfer from one to the next.

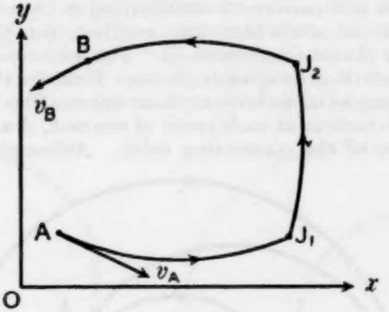

FIG. 6.

If the motion takes place in the plane of rectangular axes Ox, Oy and the x- and y-components of the gravitational attraction are $-f$, $-g$, respectively, two components (u, v) of a vector quantity called *the primer* are determined by means of the equations

$$\left.\begin{aligned} \frac{d^2u}{dt^2}+u\frac{\partial f}{\partial x}+v\frac{\partial g}{\partial x}=0, \\ \frac{d^2v}{dt^2}+u\frac{\partial f}{\partial y}+v\frac{\partial g}{\partial y}=0. \end{aligned}\right\} \quad \text{.............................(16)}$$

The quantities (u, v) vary with t along any track of the type specified. The track is optimal with respect to fuel expenditure if the quantities (u, v) are such that $u^2+v^2 \leqslant 1$ at all points on the track and satisfy the following conditions at each junction:

(i) (u, v) are continuous and represent the direction cosines of the direction of thrust,

(ii) $(\dot{u}, \dot{v})$ are continuous,

(iii) $u\dot{u}+v\dot{v}=0$.

The proof of these results will be founding in Reference 1.

Since there is no velocity loss due to the action of the finite gravitational force during an instantaneous impulsive thrust, if Δv is the magnitude of the velocity change at a junction, the ratio of the rocket mass upon arrival at the junction to the rocket mass upon departure is given by equation (3). If there

are n junctions and at the ith junction the velocity change is Δv_i and the mass ratio is r_i, then

$$\Delta v_i = c \log r_i. \qquad \text{.......(17)}$$

But, $r_1 r_2 \dots r_n$ is the ratio R of the initial mass to the final mass after the transfer has been completed. Thus

$$\sum_{i=1}^{n} \Delta v_i = c \log R, \qquad \text{.......(18)}$$

showing how the net velocity increment for the manoeuvre is related to the overall mass ratio. The net velocity increment is called the *characteristic* (or *ideal*) *velocity* of the manoeuvre.

Corresponding results which take into account aerodynamic forces have also been obtained (see Reference 2).

5. *Particular Problems.* The general results of the last Section have been employed to solve a number of elementary problems of space navigation. We shall only describe the results here.

There is the basic manoeuvre of transferring a rocket from one elliptical orbit about a centre of attraction into another, the time of transit being specified or open to choice (Reference 3). The solution takes a very simple form when the two orbits are coplanar circles. Transfer then takes place along an elliptical orbit tangential to both circular orbits (Fig. 7), impulsive thrusts being applied by the motors at each point of contact, A and B, to effect transference into and out of the connecting orbit. Although the requirement of

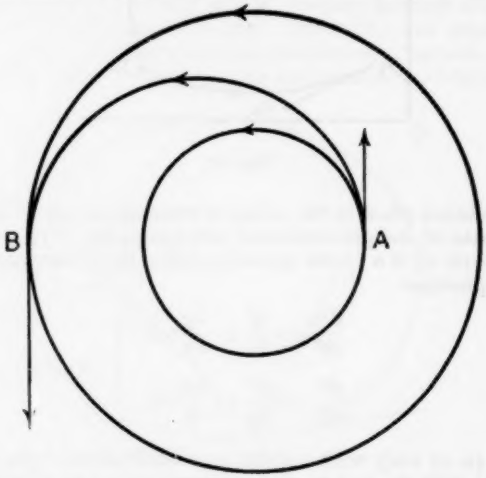

FIG. 7.

impulsive thrusts cannot be achieved in practice, there will usually be no difficulty in attaining a very close approximation. If the two orbits are those of the Earth and Mars for example, it may be calculated that the time which would be spent in the orbit of transfer would be about 260 days, as compared with two periods of acceleration of a few minutes each.

If α is the ratio of the radius of the larger to the radius of the smaller orbit, and V_1, V_2 are the respective velocities in these orbits, then the characteristic velocity for the manoeuvre may be proved to be

$$V_1\left[1-\sqrt{\frac{2}{1+\alpha}}\right]+V_2\left[\sqrt{\frac{2\alpha}{1+\alpha}}-1\right] \qquad \text{.......(19)}$$

In the case of a transfer between the orbits of the Earth and Mars, $V_1 = 24{\cdot}1$ km/sec, $V_2 = 29{\cdot}8$ km/sec, $\alpha = 1{\cdot}524$, so that the characteristic velocity for the transfer is 5·60 km/sec. With $c = 4$ km/sec, this corresponds to an overall mass ratio of 4·06.

It may be expected that a rocket ship outward bound for Mars will leave from a circular orbit about the Earth. The problem is therefore presented of calculating the most economical manoeuvre whereby a rocket may leave such a circular orbit and arrive at infinity with the velocity necessary to set it into the orbit of transfer. Assuming the thrust unlimited in magnitude, analysis reveals (Reference 4) that there are two possibilities. If the velocity required at infinity is less than a certain critical value, escape should be achieved by a single impulsive thrust tangential to the circular orbit causing the rocket to move into a hyperbolic orbit and so escape to infinity (Fig. 8*a*). If, however, the velocity at infinity is to exceed this critical value, a more complex manoeuvre saves fuel. An impulsive thrust opposing the motion is first applied to set the rocket into an elliptical orbit approaching the centre of attraction (Fig. 8*b*). At perigee a second impulsive thrust transfers the ship into a hyperbolic orbit along which it proceeds to infinity. This manoeuvre becomes more economical the closer the approach that can be made to the centre of attraction. By reversing the order of events in these manoeuvres, economical entries into circular orbits may be made with any velocity of approach.

Fig. 8*a*.

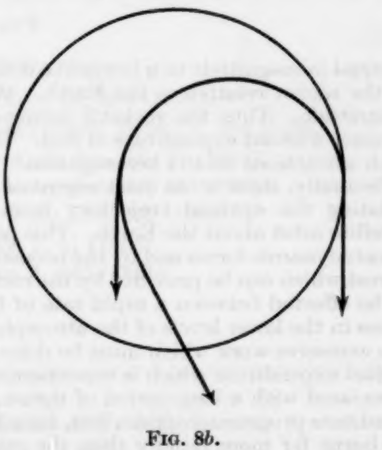

Fig. 8*b*.

It has been suggested that it might be more convenient to employ a microthrust motor operating with electrically accelerated streams of ions when escaping from circular orbit and during later manoeuvres in space (Reference 5). In this case the thrust will be limited in magnitude and impulsive thrusts cannot be approximated. The problem of escape from a circular orbit under these circumstances does not admit of a simple solution. However, equations have been found from which the optimal trajectory may be deduced (Reference 6). They require the thrust to be offset from the direction of motion towards the centre of attraction, the angle of offset being steadily reduced to zero as the manoeuvre continues.

Two other problems of a mathematical nature which have received some attention are :

Firstly, how may the attractions exercised by the various bodies of the Solar System be employed to reduce the fuel consumption for any particular

journey? Thus, a rocket bound for Mars might be arranged to pass close to the Moon so that the attraction of this body can cause an increment in the rocket's velocity. Of course, the rocket will recede from the Moon at the same speed as that with which it approaches, but, relative to the Sun or the Earth this final speed can be different from the initial speed. This fact is made clear from a study of the diagram below (Fig. 9). $\mathbf{v}_0$ is the initial rocket velocity relative to the Earth. $\mathbf{V}$ is the velocity of the Moon and $\mathbf{u}$ the velocity of the rocket relative to the Moon. The velocity of recession relative to the Moon $\mathbf{u}'$

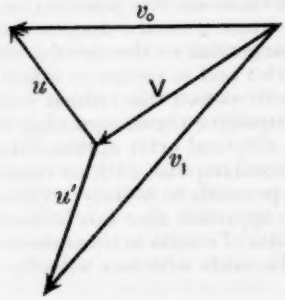

FIG. 9.

is equal in magnitude to $\mathbf{u}$ but is in a different direction. $\mathbf{v}_1$ is the final velocity of the rocket relative to the Earth. We note that $\mathbf{v}_0$ and $\mathbf{v}_1$ are unequal in magnitude. Thus the rocket's momentum has been augmented by the encounter without expenditure of fuel. The most satisfactory way of employing such attractions awaits investigation.

Secondly, there is the most urgent of all astronautical problems, that of calculating the optimal trajectory from a ground launching station into a satellite orbit about the Earth. This problem is complicated by the existence of aerodynamic forces and by the necessity for making allowance for the limited thrust which can be provided by the rocket motors. Clearly a compromise has to be effected between a rapid rate of fuel expenditure, leading to high velocities in the lower levels of the atmosphere and hence to a fuel wastage due to the excessive work which must be done against air resistance, and a slow rate of fuel expenditure which is uneconomical due to the high gravitational losses associated with a long period of thrust. It is found that an optimal fuel expenditure programme exists, but, for a large rocket, this requires that the fuel be burnt far more rapidly than the maximum rate that can be achieved in practice. We therefore assume that the motor will operate at maximum thrust and study the problem of optimising the direction of thrust. This has been done (Reference 7) and it appears that the rocket should first ascend vertically until the denser layers of the atmosphere have been cleared, when its direction of motion becomes approximately horizontal. When sufficient momentum has been acquired to carry the vehicle to the level of the circular orbit, the motors should be shut down. The rocket then coasts to the orbit along an elliptical arc tangential to the orbit and a final short burst from the motors will be necessary to effect the transfer into the satellite orbit.

6. *Conclusion.* Astronautics provides a new field for the exercise and development of mathematical techniques, especially those connected with the calculus of variations. These techniques will yield equations determining the tracks along which rocket craft are to be navigated and there then arises the further problem of computing the solution to these equations. This task may

be performed by an automatic computer but, even so, will be a formidable one. Improved methods of programming will have to be devised to permit the solution of differential equations under the complex boundary and other conditions arising.

Space travel can also be expected to influence mathematics indirectly by causing a rapid accumulation of astronomical data, all of which will have to be digested by the mathematical systems of the cosmologists. Is it probable that, with our experience confined to that available at the Earth's surface, we should discover the structure of the Universe? Direct exploration of even a small region of the galaxy will yield results greatly surpassing in significance those obtained by centuries of theorizing on inadequate data. This, of course, is certain to make astronautics unpopular with those who have already fathered a " system "!

But the ultimate effects of the influence of space travel upon mathematical development almost certainly cannot be foreseen. The view that man will take of the Earth and its inhabitants from the vantage point of outer space will differ from that prevailing today to an extent far greater than that which separates the modern and medieval outlooks. It is virtually impossible, therefore, to predict, at this early stage, many of the complex problems with which our far-voyaging descendants will be faced or the new data they will acquire and will wish to relate together in mathematical systems. We can only state with assurance that the influence upon mathematical thought of all these things will be profound.

D.F.L.

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GLEANINGS FAR AND NEAR

1884. These two last (Colin and Steve) were standing side by side and facing the younger man (Prosser) and Weaver thought that he understood. So this triangle would not be a triangle much longer he concluded. Prosser, the hypotenuse, had been squared with an overseas appointment which no doubt Steve Dennison had arranged for him, and already the other two were adjusting themselves into parallel lines.—W. Tyrer, *Such Friends are Dangerous*, p. 212. [Per Mr. L. D. Owen.]

1885. . . . . the eight-man jury returned. Asked by the coroner whether the verdict was unanimous, the foreman replied : " Ninety per cent."—*Daily Herald*, January 5, 1956. [Per Mr. A. C. Cossins.]

1886. The form of " Boléro " may be expressed mathematically as 1^{18}. The unit is an ingenious binary theme of which one or the other half is stated eighteen times without development or modulation.—David Drew *on cover of gramophone record Boléro by Ravel, Columbia 33 C 1034.* [Per Mr. J. Hooley.]

THE DEFINITION OF NUMBER*

R. L. Goodstein

It is surely a very remarkable thing that despite the range, power and success of modern mathematics, the concept of natural number, on which the whole edifice rests, is still something of a mystery.

Towards the end of the last century the great German mathematician and philosopher Gottlob Frege, dissenting strongly from the view current in his time—and perhaps still today—that number is either a physical or a mental entity, gave the first purely logical definition of a natural number. His work aroused so little interest at the time that his definition remained practically unknown until it was rediscovered by Bertrand Russell 20 years later. The definition, though a little strange at first sight, is quite simple and is based upon a notion which is of first importance in many branches of mathematics, the notion of a (1, 1) correspondence.

Frege defined number in terms of the concept " equi-numerous ". That is to say, he sought first to define the property of " having the same number ", and then to define the concept of number itself by means of this property. The reasoning which led Frege, and Russell after him, to pursue this course seems quite plain. Just as Plato sought to define the predicate red as that which all red objects have in common so Frege identified each natural number with the property common to all the classes which have this number of members. Of course, if we are to avoid a vicious circle in the definition we must define the property of having the same number of terms without reference to the actual number, and this is where the concept of (1, 1) correlation plays a decisive part. If there is a collection of cups and saucers on the table, and if there is a cup on each saucer and a saucer under each cup, then we can say that the class of cups and the class of saucers have the *same number* of members, even if we do not know what this number is. Provided, therefore, that we can define (1, 1) correlation independently of the number concept then by its means we can define the property of having the same number of terms without any reference to numbers. Leaving on one side for the moment the problem of defining (1, 1) correspondence the next stage in the Frege-Russell definition of number is the identification of a natural number n with the property of being (1, 1) related to a class of natural number n; if we make no distinction between a class and the condition for membership of the class this amounts to identifying n with the class of all classes (1, 1) related to some class of n members. There remains the task of providing sample classes for each natural number. For the number zero, we take the class whose only member is the empty class (which is defined by the property " not identical with itself "); then as a *sample* class of one member we take the class (0) whose sole member is the number zero; for the sample class of two members we take the class whose members are 0 and 1, and so on. A class is said to have 2 members for instance if it is a member of the class of classes 2, that is to say if it has the property of being (1, 1) related to the class (0, 1).

To complete the Frege-Russell definition of number we have to show that the concept of a (1, 1) relation is definable without using the notion of number. Let us first take the idea of *relation* for granted and concentrate upon the specific property of one to one relatedness.

Denoting by x, y, z variables for *things*, a generic term for the objects of our universe of discourse, and by xRy the statement that x stands in the relation R to y, the conditions for a relation R to be a (1, 1) relation are

$$xRy \,\&\, xRz \rightarrow y \equiv z, \; yRx \,\&\, zRx \rightarrow y \equiv z$$

* A lecture delivered at the Annual Meeting of the Mathematical Association, April, 1955.

As to relation itself, a relation may be defined simply as a class of ordered pairs, so that a relation R is the class of all ordered pairs (x, y) such that xRy. Finally, we may define an *ordered* pair (x, y) as the class whose only members are x, and the class whose members are x and y, so that the ordered pair is a class of two elements, one of which is distinguished by being a member of the other.

In this way the several elements in the Frege-Russell definition have all been defined in terms of the single concept of class, and there the reduction process rests with class as the sole undefined concept.

Apart from the logical connectives and, or, not implies, the only connectives introduced in the definition of number are class membership and identity, and in fact the second of these may be defined in terms of the first.

For instance we may take

$$(a \equiv b) \longleftrightarrow (\forall x)(a \in x \longleftrightarrow b \in x)$$

where " $\longleftrightarrow$ " is the logical connective " implies and is implied by ", $(\forall x)$ is a prefix denoting universality, and " $a \in x$ " reads " a is a member of x ". The definition of number involves therefore only a single concept and a single binary connective.

The Frege-Russell definition, like a great part of modern mathematics, can only be expressed in a logic capable of making universal statements about classes. Class logic, as we may call it, has however the doubtful distinction of being perhaps the most spectacularly unfortunate of man's creations. Frege's system of class logic was found to be self-contradictory just as his second volume was passing through the press. It was a bitter blow to a man who had received little enough encouragement from his contemporaries, and as he himself observed, small consolation to him that the contradiction was not peculiar to his system but was common to all work on set theory current in his time. The contradiction arises when we consider the class α whose members are those classes which are not members of themselves. By definition, x belongs to α if and only if x is not its own member, i.e.

$$(x \in \alpha) \longleftrightarrow x \notin x.$$

Taking α for x we obtain the blank contradiction

$$(\alpha \in \alpha) \longleftrightarrow (\alpha \notin \alpha).$$

Many unsuccessful attempts have been made to formulate a demonstrably consistent class logic. Russell's system has the safeguard of a hierarchy of types which places a class in a type above that of its members and bans the formation of classes whose members do not all belong to the same type ; one of the disadvantages of this system is that it has to postulate the existence of an infinite set to ensure that there is no greatest number. Another is that it furnishes us with not one sort of natural number, but infinitely many sorts, one in each type. A system of Quine's avoided this reduplication of the natural numbers at the cost of making it impossible to prove there are n numbers from 1 to n ; this system has recently been shown to contradict the axiom of choice. Another system, due to Zermelo seeks to keep out of trouble by restricting the membership of new classes to members of existing classes, making the generation of new classes rather uncertain and laborious. Von Neumann and Bernays have formulated systems in which certain classes only are distinguished as elements and only these distinguished classes are allowed to be members of classes. A later system of Quine's which also introduced a condition of element-hood was shown to be self-contradictory on the eve of publication. Quine's system was repaired by Hao Wang who has just recently produced a set theory of his own which admits a transfinite series of levels.

In view of the dangers to which class logic is exposed it is obviously desirable

to find a treatment of the natural numbers which can be formulated in a more economical and relatively safer level of logic. Of course if the concept of class was intrinsic to every branch of mathematics there would be no point in seeking for a class-free definition of number, but in fact there are interesting and important branches of mathematics, for instance the classical theory of numbers and axiomatic projective geometry, in which the class concept is totally dispensable.

It is actually quite easy, following Peano, to give an account of arithmetic in which number is taken as a primitive concept. If we denote number variables by a, b, c and take S, 0, $=$ as primitive constants (where " S " may be interpreted as " successor of ") then arithmetic may be based on the axioms

$$a=b \longleftrightarrow Sa=Sb \qquad Sa \neq 0$$

$$(a=b) \rightarrow ((a=c) \rightarrow (b=c))$$

$$a+0=a, \quad a+Sb=S(a+b)$$

$$a\,.\,0=0, \quad a\,.\,Sb=a\,.\,b+a$$

and the induction axiom

$$\{P0 \;\&\; (\forall a)(Pa \rightarrow PSa)\} \rightarrow Pa$$

With the aid of a suitable system of logic (quantification theory with number variables only) these axioms suffice for the proof of all the familar theorems of elementary arithmetic. We can in fact dispense with logic entirely (and with the first three axioms) if we make provision for the definition of new functions. However the question in which we are now interested is the definition of number. To what extent can these axioms be said to constitute a definition of number?

What the axioms in fact provide are transformation rules for certain patterns. For instance they enable us to pass from the pattern $S0+SS0$ in turn through $S(S0+S0)$, $SS(S0+0)$ to $SSS0$. If we now introduce the familiar abbreviations $1=S0$, $2=1+1$, $3=2+1$ and so on, then we are able to make the passage from $1+2$ to 3. But what has this to do with numbers? Are numbers no more than shapes, like " 2 " or " 5 ". This is obviously not the case since numbers have such properties as being odd or even, whereas patterns like " 2 " or " 5 " are printed in ink, or painted on the gate. Are there perhaps no such things as numbers, only numerals?

Instead of pursuing the question whether numbers *exist* let us turn for a moment from mathematics to chess to consider the existence of the king of chess. Shall we say that the king of chess exists or not? Obviously the king of chess is not simply the particular piece of wood or ivory which is moved about on the chess board. For one thing the game could be played with a cork in place of the king, or for that matter the king and queen pieces could be interchanged. Now what does interchanging the king and queen consist in? It does not involve any alteration to the pieces themselves, but only to the moves which they make. It follows that what constitutes a piece king are not its shape or size but the kind of moves it makes. A piece is king because it makes the king's moves in the game. Thus we can say that the king of chess is one of the parts which a piece plays in the game of chess, and we can characterise this part by describing the moves.

The king of chess is a role in a game in precisely the way that the King of Ruritania is a role in a play. And so too the natural numbers are roles in arithmetic, the number two for instance being the role which arithmetic assigns to the numeral 2, and so on.

Russell and Frege both opposed the use of number variables as primitive signs on the grounds that numbers belong, not only to arithmetic but also to

everyday life. Thus we use numbers not only in such contexts as $2+3=5$ but in shopping and counting planets. On the basis of the Frege-Russell definition we can prove in class logic such assertions as:

"If A, B, C are white then three things are white,"

and such statements as this were held to be incapable of proof if numbers were taken as primitive symbols. However, nothing more is needed than the introduction into arithmetic of a suitable counting operator. If x is a variable for names $A, B, C, \ldots$ and ξ is a variable for classes of names like A & B & C, then we may define a counting operator N by the recursive equations

$$N(x)=1$$
$$N(\xi \;\&\; x)=N(\xi)+1,$$

From these equations we may derive in turn

$$N(A)=1,\ N(A \;\&\; B)=N(A)+1=1+1=2,$$
$$N(A \;\&\; B \;\&\; C)=N(A \;\&\; B)+1=2+1=3, \quad \text{and so on.}$$

Although formalised arithmetic is adequate both for mathematics and logic it is an extraordinary fact that no formalisation of arithmetic is adequate to characterise the number concept completely. Every axiomatic system admits a valid interpretation in which a richer class of elements than the natural numbers plays the number role, and this is equally true whether we take numbers as primitive symbols or define them in class logic as in the Frege-Russell definition.

I am going to turn now from natural numbers to consider other number systems.

Integers

The passage from natural numbers to integers presents no difficulty. Probably the simplest course is to define an integer as an ordered pair of natural numbers $[a, b]$ with an arithmetic

$$[a, b] \gtreqless [c, d] \longleftrightarrow a+d \gtreqless b+c$$

$$[a, b]+[c, d]=[a+c, b+d],\quad [a, b]-[c, d]=[a+d, b+c],$$
$$[a, b]\,.\,[c, d]=[ac+bd, ad+bc].$$

It readily follows that $\{[a, b]+[c, d]\}-[c, d]=[a, b]$

$$\{[a, b]-[c, d]\}+[c, d]=[a, b],$$

and, most important, that

$$\{[a, b]=[a', b'] \;\&\; [c, d]=[c', d']\}$$
$$\rightarrow\{[a, b]\pm[c, d]=[a', b']\pm[c', d'] \;\&\; [a, b]\,.\,[c, d]=[a', b']\,.\,[c', d']\}.$$

If $a \geqslant b$, $[a, b]=[a-b, 0]$ and if $a<b$, $[a, b]=[0, b-a]$ so that every integer is expressible in one of the forms $[x, 0]$, $[0, x]$.

Writing $+x$ and $-x$ for $[x, 0]$, $[0, x]$ respectively we regain the familiar representation of integers by $+$ and $-$ signs. The so-called rules of sign for integers are of course provable consequences of the defining equations for addition, subtraction and multiplication. For instance the rule "minus times minus makes plus" is proved by showing

$$(-x)\,.\,(-y)=[0, x]\,.\,[0, y]=[xy, 0]=+xy.$$

All this is of course well known. It is, however, a not uncommon fallacy to suppose that for $+x$ we must take, not $[x, 0]$ as we have done but the class of

all pairs equal to $[x, 0]$, i.e. all the pairs $[x+n, n]$, presumably on the grounds that we must not identify an integer with one of its representations. Since a concept is in fact no more to be identified with the *class* of its representations than with any one of them this re-introduction of the logic of classes into the theory of integers is without point.

The arithmetic of fractions, like the arithmetic of integers, is simply an arithmetic of ordered pairs of natural numbers p/q. Of rather more interest is the arithmetic of pairs of *integers* ξ/η, $\eta \neq 0$, with the rules

$$\xi/\eta \gtreqless \xi'/\eta' \longleftrightarrow \xi\eta\eta'^2 \gtreqless \xi'\eta'\eta^2$$

$$\xi/\eta \pm \xi'/\eta' = (\xi\eta' \pm \xi'\eta)/\eta\eta'$$

$$(\xi/\eta) \,.\, (\xi'/\eta') = \xi\xi'/\eta\eta'$$

and

$$(\xi/\eta) \div (\xi'/\eta') = \xi\eta'/\xi'\eta, \quad \xi' \neq 0.$$

If these rules are fully expressed in terms of natural numbers they become rather formidable. At the price of a little awkwardness over division a simpler theory is obtained by taking rational numbers as triples of natural numbers $(p, q)/r$, $r>0$. For equality and inequality

$$(p, q)/r \gtreqless (p', q')/r' \longleftrightarrow (pr', qr') \gtreqless (p'r, q'r).$$

Multiplication has the expected form

$$(p, q)/r \times (p', q')/r' = (pp'+qq', pq'+p'q)/rr'$$

and for division we may take

$$p' \neq q' \rightarrow (p, q)/r \div (p', q')/r' = (p, q)/r \times (p'r', q'r')/(p'-q')^2.$$

Complex numbers

The familiar definition of complex numbers as ordered pairs of real numbers with postulated laws of equality, addition and multiplication, as set out for instance in the Associations last Trigonometry Report, is logically sound, economical and effective. It is sometimes criticised however from the pedagogic standpoint for its failure to indicate its motivation. Certainly the multiplication law is a strange one and its connection with $\sqrt{-1}$ by no means immediately evident, but I think this difficulty is intrinsic since every logical development of complex arithmetic must either postulate a number whose square is -1 or assure the existence of such a number by some devious route.

One such route is to define complex numbers as the residue class, modulo ξ^2+1, of the field of real numbers extended by an indeterminate ξ. The residues modulo ξ^2+1 are the binomials $a+b\xi$ where a, b are real numbers, and ξ satisfies the equation $\xi^2=-1$ in virtue of the congruence

$$\xi^2 \equiv -1 \,(\mathrm{mod}\ \xi^2+1).$$

Taking residues modulo ξ^2+1 avoids postulating the existence of a number whose square is -1 but it is by no means self-evident to the uninitiated that the residue class mod(ξ^2+1) is a number field.

There are however more serious obstacles in the path of this development of complex arithmetic. The extension of the field of real numbers by an indeterminate ξ is defined to be the ring of polynomials

$$a_0 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers. This ring in turn may be defined as the class of all ordered sets of real numbers

$$(a_0, a_1, a_2, \ldots, a_n),\ n = 1, 2, 3, \ldots$$

with postulated equality, addition and multiplication laws

$$(a_0, a_1, \dots, a_n) = (b_0, b_1 \dots, b_n), \quad \text{if and only if } a_r = b_r,\ r = 1, 2, \dots, n,$$

$$(a_0, a_1, \dots, a_n) + (b_0, b_1, \dots, b_n) = (a_0 + b_0, \dots, a_n + b_n)$$

$$(a_0, a_1, \dots, a_n) \,.\, (b_0, b_1, \dots, b_n) = (a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + a_1 b_1,\ a_2 b_0 \dots).$$

If the polynomial ring is defined in this way the definition of complex numbers as the residue class modulo (1, 1) can no longer be preferred to the definition in terms of ordered pairs, since it verges on the absurd to replace a definition in terms of ordered pairs by a definition in terms of ordered sets of arbitrary length.

The polynomial ring can however be defined without introducing the concept of an ordered set. We introduce two primitives + and ., a polynomial ξ and variables for polynomials $a, b, c, \dots$ and lay down the axioms

$$a + b = b + a \qquad (a + b) + c = a + (b + c)$$

$$a \,.\, b = b \,.\, a \qquad (a \,.\, b) \,.\, c = a \,.\, (b \,.\, c)$$

$$(a + b) \,.\, c = a \,.\, c + b \,.\, c.$$

The concept "polynomial of degree n" is defined inductively as follows: Real numbers are polynomials of degree zero; ξ is a polynomial of degree unity; if a and b are polynomials of degrees m and n respectively then $a + b$ is a polynomial of degree $\max(m, n)$ and $a \,.\, b$ is a polynomial of degree $m + n$. For equality between polynomials we postulate the inference rule:

If α, β are polynomials of degree zero and if

$$\alpha + a\xi = \beta + b\xi$$

then $\alpha = \beta$ and $a = b$.

By induction over the degree of the polynomial we can prove that every polynomial of the system is expressible in the form

$$a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \dots + a_n\xi^n$$

where ξ^n is an abbreviation for the product $\xi \,.\, \xi \dots \xi$ with n factors, showing that the elements of the axiom system are the polynomials of the ring. Apart from the equality rule, the axioms are all of the kind familiar in elementary arithmetic, and as a definition of the polynomial ring could hardly be simpler, but as an introduction to complex arithmetic it is a formidable journey to make to avoid the postulation of the multiplication law for ordered pairs.

Real numbers

In talking about complex numbers I have taken the real numbers for granted and it is to them that I must now turn. The logical foundation of the theory of real numbers presents formidable difficulties. The statement and proof of *Dedekind's Theorem* for instance requires of course a fully fledged class logic for its formalisation. Along with the complexities and perils of a class logic we have two further bitter draughts to swallow. The first is that *no* formal system is rich enough to *define* all real numbers, because all the statements of a formal system can be enumerated in a simple sequence, and therefore at most a denumerable infinity of real numbers can be *defined.* And, definition apart, Cantor's proof of the non-denumerability of the class of real numbers is now known to establish only *relative* non-denumerability. For every formalised theory of real numbers can be shown to have a denumerable model; that is to say there is an interpretation of the predicates of the system under which every true statement about classes of integers becomes a true statement about integers. Since the integers are denumerable it follows that the class of real numbers in the system is denumerable.

Cantor's proof of non-denumerability is therefore a proof of the incompleteness of the formal system, a proof that the function which is known to enumerate the class of real numbers is undefinable in the system. Even without appeal to the known existence of a denumerable model the relativity of Cantor's proof is apparent. For the proof starts by saying : " let $a_1, a_2, \ldots$ be an enumeration of the real numbers " and this statement of course has a concealed existential premiss " If the class of real numbers is denumerable in our formalism ". The conclusion to be drawn from the familiar *reductio ad absurdum* proof is therefore that the class is not denumerable in our formalism. To pass from this to absolute non-denumerability we must know that our system of class logic is complete, i.e. that *all* enumerations are definable in it, and precisely *this* proves not to be the case.

If we are content to formalise only a constructive fragment of real number theory we can dispense with class logic entirely and use restricted predicate logic or simply free variable logic.

In free variable logic however real numbers lose most of their familiar properties. Let $r_1, r_2 \ldots$ be one of the familiar enumerations of the rationals, and let us, following Dedekind, define a recursive real number to be a recursive predicate $P(n)$ such that there is no greatest r_n for which $P(n)$ holds and if $r_m < r_n$ then $P(n)$ entails $P(m)$. Further, let us say that a recursive sequence of rationals $s(n)$ is *recursive* convergent if there is a recursive n_k so that

$$n \geqslant n_k \rightarrow |\, s(n) - s(n_k) \,| < 10^{-k}.$$

Next we define a recursive decimal to be $\sum\limits_{r=0}^{\infty} f(n)\, 10^{-n}$ with $f(n)$ recursive and $0 \leqslant f(n) < 10$ for $n \geqslant 1$, and a recursive nest of intervals to be a pair of recursive sequences of rationals a_n, b_n such that

$$a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n$$

and

$$n \geqslant n_k \rightarrow b_n - a_n < 10^{-k}.$$

If in these definitions we take recursive functions to be *primitive* recursive functions, the functions of elementary arithmetic, then none of the expected connections holds. It can be shown that there are recursive convergent sequences which have no recursive real limit, recursive real numbers which are not recursive decimals and recursive nests which do not contain a recursive real number.

If we interpret recursive in the much wider sense of general recursive, and use predicate logic then *some* of the expected connections can be established, but we are as far away as ever from classical analysis. There are recursive real numbers which can neither be proved equal, nor proved unequal ; recursive real numbers which cannot be proved rational nor proved irrational and monotonic increasing and bounded sequences which have no recursive limit. In fact the function theory of a constructive fragment of the class of real numbers is not significantly richer than the system of analysis which can be developed in a rational field.

R. L. G.

1887. Si on peut tenir dans une petite allégorie les activités du mathématicien, nous le comparerons volontiers à un touriste revenant d'une série de voyages fortunés. Il a traversé des édens. Il en est ivre. Il s'abandonne à des transports, dans lesquels l'égocentrisme où l'immodestie ne sont pour rien.—Pierre Anspach, *Aperçu de la Théorie des Polygones Réguliers.* [Per Mr. H. M. Cundy.]

SUMS OF POWERS OF THE NATURAL NUMBERS

By Sheila M. Edmonds

When one first comes across the formulae

$$1+2+\ldots+N=\tfrac{1}{2}N(N+1), \qquad\qquad (1)$$

$$1^3+2^3+\ldots+N^3=\tfrac{1}{4}N^2(N+1)^2, \qquad\qquad (2)$$

one is immediately struck by the fact that the second sum is the square of the first. The usual methods of proof give no reason to expect that the sums will be related in such a pleasant and easily memorable way ; one is left with the impression that this is a happy accident. The following proof may give some entertainment because it throws light on this point ; it emerged unexpectedly out of the tortuous process of making up a question for a recent Scholarship examination.

Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be any sequences, and let $A_n=a_1+a_2+\ldots a_n$, $B_n=b_1+b_2\ldots+b_n$. Since $a_n=A_n-A_{n-1}$, and similarly for b_n, we have at once

$$\sum_{n=1}^{N}[a_nB_n+b_nA_{n-1}]=A_NB_N \qquad\qquad (3)$$

(where A_{n-1} is taken as zero for $n=1$). This is, of course, simply the familiar formula for " partial summation " (analogous to integration by parts).

First, take $a_n=b_n=1$. Then $A_n=B_n=n$, and (3) becomes

$$\sum_{n=1}^{N}[n+(n-1)]=N^2.$$

Thus

$$2\left(\sum_{n=1}^{N} n\right)-N=N^2,$$

which proves (1); there are, of course, simpler methods in this case.

Next, take $a_n=b_n=n$. Then (1) gives A_n and B_n, and (3) becomes

$$\sum_{n=1}^{N} n[\tfrac{1}{2}n(n+1)+\tfrac{1}{2}(n-1)n]=A_N^2,$$

that is,

$$\sum_{n=1}^{N} n^3=\left(\sum_{n=1}^{N} n\right)^2 \qquad\qquad (4)$$

$$=[\tfrac{1}{2}N(N+1)]^2.$$

It is natural to ask whether the method gives interesting results when other powers of n are chosen for a_n and b_n. If we take $a_n=1$, $b_n=n$ in (3), we get

$$\sum_{n=1}^{N}[\tfrac{3}{2}n^2-\tfrac{1}{2}n]=\tfrac{1}{2}N^2(N+1),$$

and the formula for Σn^2 follows from this and (1). In theory we could sum $\sum_{n=1}^{N} n^k$ in this way for any value of the positive integer k, provided the sums for smaller values of k were known ; but the method would clearly be less systematic than those in use. Again, we might hope to obtain other results analogous to (4) by taking $a_n=b_n=n^k$. For instance, having calculated Σn^2

we are naturally led to try $a_n = b_n = n^3$ in (3), since (2) gives A_n and B_n. We obtain

$$\tfrac{1}{2}\sum_{n=1}^{N}[n^7+n^5] = \left(\sum_{n=1}^{N} n^3\right)^2$$

$$= \left(\sum_{n=1}^{N} n\right)^4 = [\tfrac{1}{2}N(N+1)]^4.$$

The right-hand sides here have the kind of form we are seeking; but unfortunately the left-hand side is less simple than that of (4), as the expression to be summed does not now consist of a single term only. As a matter of fact, no higher power of n would have served us better; there are no other integer powers of n for which the corresponding sums are related as in (4).

If

$$\sum_{n=1}^{N} Cn^p = \left(\sum_{n=1}^{N} n^q\right)^2$$

(identically in N), where p and q are positive integers and C is a constant, then $p=3$, $q=1$, $C=1$.

Proof. It is obvious from the case $N=1$ that $C=1$. Moreover, for any positive integer k, $\sum_{n=1}^{N} n^k$ is a polynomial of degree $(k+1)$; hence $p=2q+1$.

Suppose, then, that

$$\sum_{n=1}^{N} n^{2q+1} = \left(\sum_{n=1}^{N} n^q\right)^2. \qquad (5)$$

Take $a_n = b_n = n^q$ in (3). Then, by (3) and (5),

$$\sum_{n=1}^{N} n^q(A_n + A_{n-1}) = A_N^2 = \sum_{n=1}^{N} n^{2q+1}. \qquad (6)$$

Since this is true for all values of N, corresponding terms of the two sums in (6) must be equal (the Nth term being $\sum_{n=1}^{N} - \sum_{n=1}^{N-1}$ in each case); and so

$$A_n + A_{n-1} = n^{q+1}.$$

Since also

$$A_n - A_{n-1} = n^q,$$

it follows that

$$A_n = \tfrac{1}{2}n^q(n+1), \quad A_{n-1} = \tfrac{1}{2}n^q(n-1).$$

Replacing n by $n-1$ in the first of these equations, and comparing with the second, we get

$$(n-1)^q n = 2A_{n-1} = n^q(n-1),$$

and so $q=1$. This is the required result.

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Two recent notes have drawn attention to other ways of making the relationship between sums of cubes and sums of first powers more intuitively obvious; note 2678 (by A. N. Nicholson) and note 2687 (by R. F. Wheeler).

STELLATIONS OF THE RHOMBIC DODECAHEDRON

BY DORMAN LUKE

The rhombic dodecahedron has been described by Hope-Jones, *Math. Gazette*, Vol. XX, p. 254 (1936); it can be obtained simply by taking a cube, joining each vertex to the centre, thus obtaining six square pyramids, and placing these pyramids on the faces of a second cube. The solid thus constructed is shown in fig. 1.

If the planes of the faces are produced indefinitely, their intersections with any one face are shown in fig. 8A, which contains 10 lines of intersection, one for each face, excluding the face itself and the parallel face. The region " 0 " is the original face. The regions " 0 " and " 1 " together form a connected solid, the FIRST STELLATION (fig. 2). When regions " 2 " are added, the SECOND STELLATION (fig. 3) results, and the regions " 3 " complete the THIRD STELLATION (fig. 4) and bring the series to a close. If however the second stellation is removed bodily from the third, a connected solid again results, having the areas 3 and the " undersides " of the areas 2 on its faces. This is shown in fig. 5. The original rhombic dodecahedron can be added to this, producing fig. 6. The faces of these last two polyhedra consist of a number of areas connected only at vertices: all these areas are congruent to the original rhombus " 0 ", or to a triangle which is half that rhombus cut off by the short diagonal.

It should be clear from fig. 6 that the rhombic dodecahedron is the solid common to three intersecting, mutually perpendicular, square prisms. These prisms are shown in fig. 7, produced until their square faces form part of a truncated octahedron, shown in outline only. The original rhombic dodecahedron appears inside, and can be seen down the square tunnels.

Nets for the construction of these stellations are given in figs. 8–10. Begin by constructing the rhombic dodecahedron. 12 pyramids made from the net of fig. 8B added to its faces, will give the first stellation. (Barred lines are to be scored *on the back*). 8 pieces made from fig. 9 will convert this into the second stellation, and finally 6 pieces " A " (fig. 10) will add the ends of the three square prisms. 6 pieces " B " (fig. 10) will then close in the ends of these prisms and form the third stellation. These pieces " B " should be added last, to facilitate reaching the tabs to stick together the earlier stages.

The construction of the polyhedra shown in the other two figures is left to

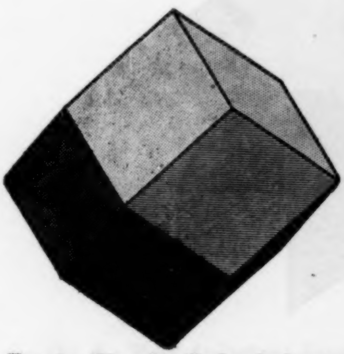


FIG. 1. Rhombic Dodecahedron.

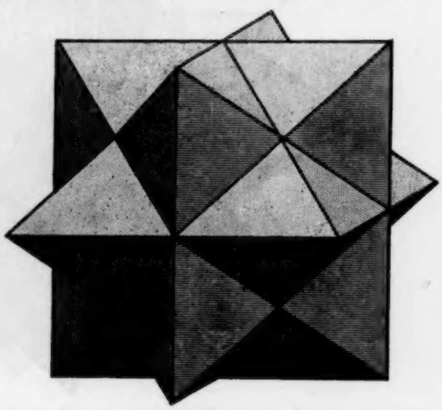


FIG. 2. First Stellation.

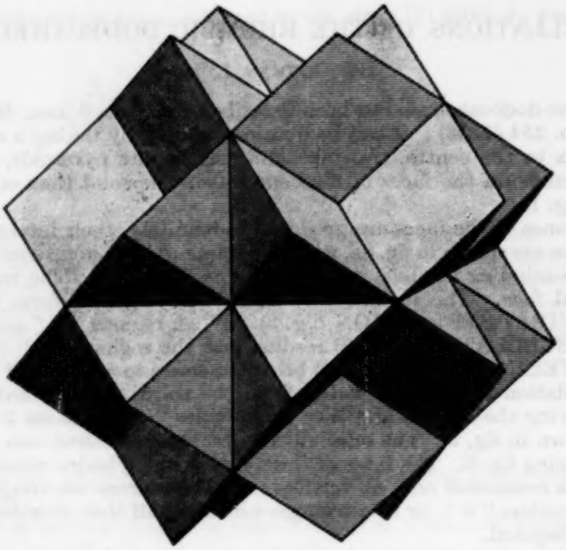


FIG. 3. Second Stellation.

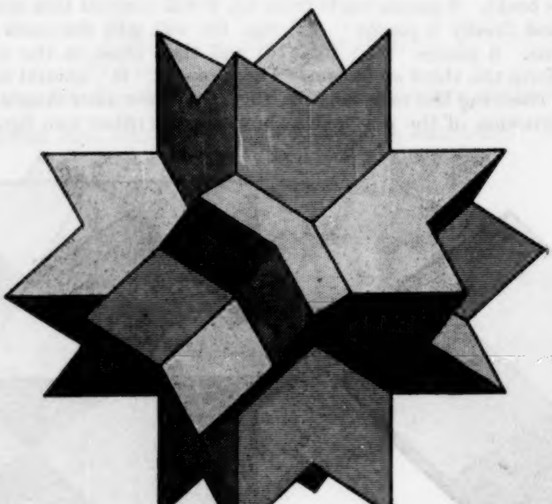


FIG. 4. Third Stellation.

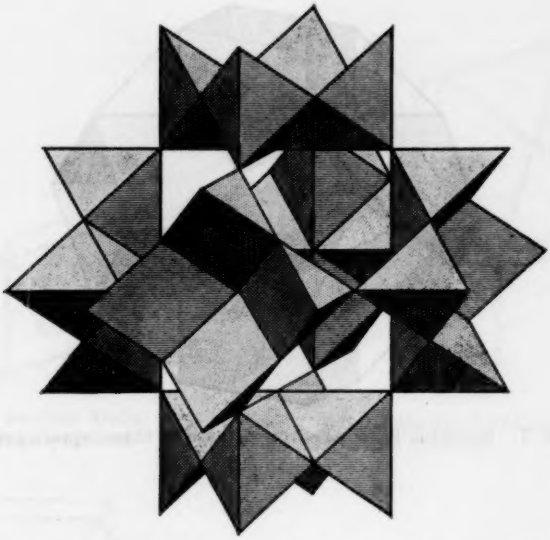


FIG. 5. Third Stellation with second stellation removed.

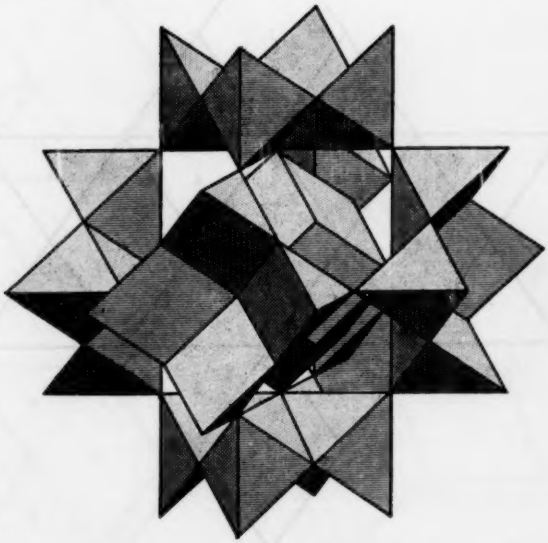


FIG. 6. Third Stellation – Second Stellation + original Rhombic Dodecahedron.

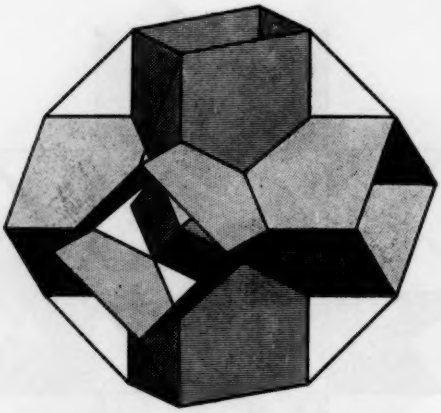


FIG. 7. Rhombic Dodecahedron encased in three square prisms.

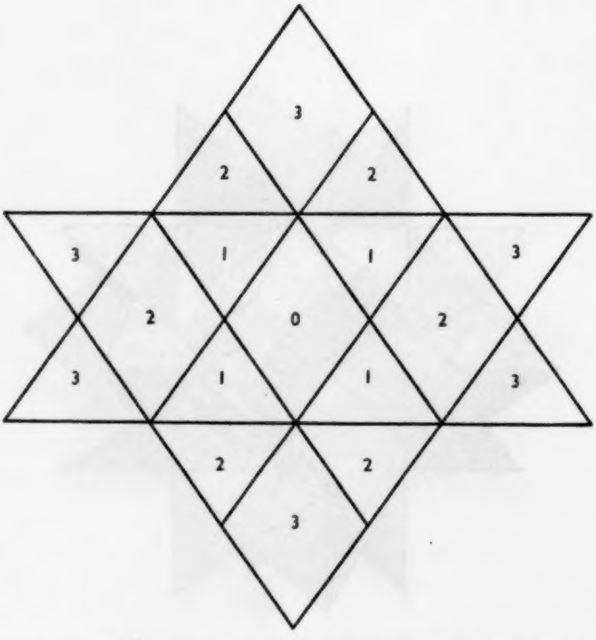


FIG. 8A. Key Diagram for Stellations.

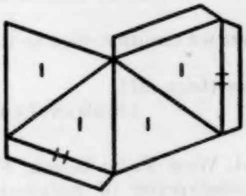


FIG. 8B. Net for First Stellation. 12 of these required.

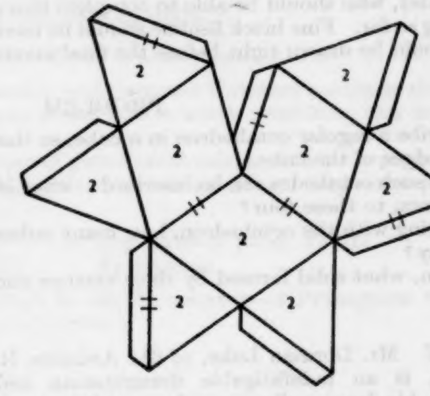


FIG. 9. Net for Second Stellation. 8 of these required.

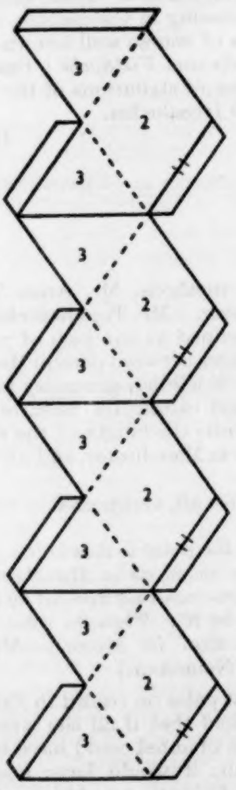


FIG. 10A. 6 required.

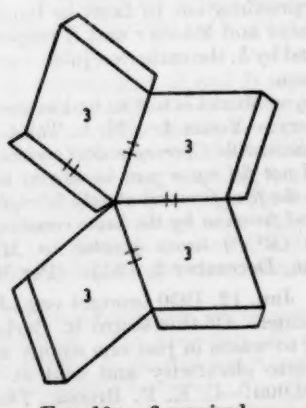


FIG. 10B. 6 required.

the reader, who should be able to complete them without difficulty after progressing so far. Fine black fishline should be used to join the sections together, and should be drawn tight before the final inverted pyramids are added.

PROBLEM

Inscribe a regular octahedron in a cube, so that its vertices are one on each of six edges of the cube.

Four such octahedra can be inscribed ; what is the solid with 32 faces which is common to these four?

Starting with the octahedron, how many cubes can we circumscribe to it in this way?

Again, what solid formed by their vertices encases them all?

DORMAN LUKE.

(*NOTE.* Mr. Dorman Luke, of 617 Ardmore Road, West Palm Beach, Fla. U.S.A., is an indefatigable draughtsman and constructor of polyhedra. Among his large collection of unpublished drawings, many of which are beautiful and intricate, there are a number of original figures and constructions. These are for the most part too extensive to publish in the *Gazette*, but he would welcome correspondence with interested readers. Those given above form a set which are simple to construct, interesting in themselves, and may whet the appetite for more. Figs. 1 and 2 are of course well-known ; figs. 3 and 4 appear in Brückner's monumental *Vielecke und Vielfläche* ; figs. 5 and 6 are believed to be original, and complete the set of stellations of the rhombic dodecahedron along the lines of the famous 59 Icosahedra.

H. M. C.)

1888. DIMENSIONS OF PANS.

To the Editor of the Manchester Guardian

Sir,—In criticising Joyce Muriel's cookery methods, Mr. Aram Topalian evokes " first principles," but fails to apply them. Mr. Topalian claims that the correct balance between rice and juice—defined as one inch of juice over rice-level—will be upset by dimensional differences between pans in Manchester and pans in Mecca. Assuming, however, that Euclidian geometry supplies a good approximation to facts in both cities, and calling the radii of pans in Manchester and Mecca r and R respectively, while the height of the rice-layer is denoted by h, the ratio rice/juice $=\pi r^2 h/\pi r^2 = h$ in Manchester, and $\pi R^2 h/\pi R^2 = h$ in Mecca.

So Joyce Muriel is not so bad at geometry after all, and probably better still at cookery.—Yours &c., M. L. Telcs, Ph.D.

[*Our Scientific Correspondent writes : Surely the point is that half a pound of rice will not fill up a pan in Mecca to the same height as in Manchester. The " h " in the first fraction should be replaced by some constant divided by r^2 and in the second fraction by the same constant divided by R^2. Then the ratio of rice to juice is (R^2/r^2) times greater in Manchester than in Mecca.*]—*Manchester Guardian*, December 9, 1955. [Per Mr. B. H. Neumann.]

1889. Jan. 12, 1930 brought one of the worst gales on record in England to the Channel. Of this storm R. Corless remarked that if all the wind energy running to waste in just one square mile of the Channel could have been converted into electricity and sold at ½d. a unit, it would have been worth £750,000,000!—C. E. P. Brooks, *The English Climate*, pp. 144–5. [Per Mr. P. H. Maby.]

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,—I teach scholarship and Common Entrance maths at the above prep. school and wonder if you would find it worth publishing this true little story.

My top form were doing a set of geometrical riders and I have rammed into them that whenever they make a statement in geometry, they must qualify it with a reason ; for example :

$AB = CD$ opp. sides of parm $ABCD$
$AB^2 = BC^2 + AC^2$ Pythagoras.

In one of the riders they had to use the converse of Pythagoras to prove the final part, and one little boy wrote :

$AB^2 = 5$
$AC^2 + BC^2 = 4 + 1$
$= 5$
therefore $\angle ACB$ is a right angle. SAROGAHTYP

Yours etc., J. K. PRIESTMAN

Orley Farm, Harrow-on-the-Hill, Middlesex.

To the Editor of the *Mathematical Gazette*

DEAR SIR,—In replying to the letter of Mr. D. F. Lawden about the laws of friction in the *Mathematical Gazette* of May 1957, I should like to make a few general points.

The version of the laws of friction quoted by Mr. Lawden :

(1) The reaction must lie within the cone of friction, A.
(2) The friction opposes the tendency to relative motion,

represents the view accepted in books by Routh, Loney, etc. Also current is a version which I now paraphrase from *Lamb's Statics*, p. 62 :

The configuration of a body which touches a rough surface at points $P_1, \ldots, P_n$ is an equilibrium configuration if, and only if, there exist forces of reaction $R_1, \ldots, R_n$ such that (i) the statical conditions are fulfilled, and (ii) R_i does not lie outside the cone of friction at P_i. B.

Now although A itself is not known to be inconsistent, A and B together are shown to be inconsistent in Note 2510 (*Mathematical Gazette*, May 1955), by the example of a rod resting obliquely against a wall. It seems reasonable to express the matter in this way since, for example, Loney's treatment in § 174 of his *Statics* of a rod resting obliquely over a wall suggests that A and B can be applied according to convenience and that therefore they are surely not inconsistent. Moreover the conjecture that A and B are equivalent is supported by most soluble friction problems including, as is shown in Note 2606 (*Mathematical Gazette*, May 1956), those involving a rigid body which touches a rough plane at a number of points.

Law B states a rule whereby the set E of *equilibrium configurations* of a specified dynamical system may be predicted. The boundary E' of E will comprise the *limiting equilibrium configurations*. In general law A may predict a different set F of equilibrium configuration, with boundary F'. Because of (1) F is contained in E. Rule (2) is open to different interpretations. Thus

according to Ramsey, but not Loney, the friction forces must oppose what would be the initial motion of the system if everything were smooth. But one might demand merely that the friction forces oppose some one *possible* motion of the system. For an elastic body any motion is possible and one would find $F = E$. One might even allow non-rigid motions when rigid bodies are involved, because they cannot, physically, be ideally rigid. In any case the various interpretations of (2) seem to aim at predicting the set of limiting equilibrium configurations, and before such a theory is acceptable it must be shown that this set bounds. Further, although version A should be generalised to deal with anisotropic friction, there seems to be no obvious way of doing this.

It can scarcely be denied that Law B is simpler than Law A, and Lamb considered it to be the natural generalisation of the (undisputed) laws of friction for a particle. Moreover version A does not correspond to a unique clearly defined general theory, and whether it can be modified so as to do so seems to me to be open. Still it is of course important to make experiments in special cases where A and B lead to different predictions. Naturally, in experimental work, the *stability* of the equilibrium configurations is important. Those which are nearly limiting will tend to be unstable because the dynamic coefficient of friction is usually less than the static. I have heard that some rough experiments initiated by Professor S. Goldstein at Harvard on a rod resting against a wall seem to support version B.

Yours etc., T. A. S. Jackson

University of Liverpool

To the Editor of the *Mathematical Gazette*

Dear Sir,—The sentence " One might ask . . . section " in my review on p. 78 of the last issue of the *Gazette* is of course nonsensical and is due to carelessness on my part. My intention was to point out that the reader is not told that integration is with respect to t and that a new symbol p is introduced without any explanation of its significance. The exposition is of course intelligible and unnecessary to anyone who knows what a Laplace Transform is. The beginner who does not know will, in my opinion, find the explanation in the book inadequate.

Yours etc., D. E. Rutherford

Department of Mathematics, St. Salvator's College, St. Andrews

To the Editor of the *Mathematical Gazette*

Dear Sir,—I am presently a sophomore attending Rensselaer Polytechnic Institute in Troy, New York. While reading a copy of the *Mathematical Gazette* from December 1903 (Vol. II, No. 42), I came across an error in the third problem which I would like to call to your attention, if it has not already been noticed.

The problem reads as follows :

Shew that the result of eliminating x, y, z from

$$\begin{gathered} x+y+z=a \\ (y-z)^2+(z-x)^2+(x-y)^2=b \\ x(y-z)^2+y(z-x)^2+z(x-y)^2=c \\ x^2(y-z^2)+y^2(z-x^2)+z^2(x-y^2)=d \end{gathered}$$

is

$$b^2-2a^2b+12ac-18d=0.$$

The fourth equation, giving the value of d, should read

$$x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2=d,$$

in order for the solution to be valid.

As a matter of interest I also eliminated x, y, z from the original four equations the result being:

$$4a^4-12a^2-8ba^2+6ba+24ac-18c+3b^2+108d \\ +3\sqrt{3}\sqrt{(2a^3-6c-ab)(2a^3+6c-5ab)-(2a^2-b)^2(a^2-2b)}=0.$$

My method of solution was to employ the identity

$$xy^2+yz^2+zx^2-x^2y-y^2z-z^2x=(x-y)(y-z)(z-x),$$

square both members of this identity, and evaluate the left-hand side in terms of a, b, and c.

Yours, etc., Peter D. Zvengrowski

Rensselaer Polytechnic Institute

To the Editor of the *Mathematical Gazette*

Dear Sir,—I should like to make a few remarks in connection with Mr. B. C. Brookes' excellent review of *Théorie mathématique du Bridge à la portée de tous* by Borel and Chéron.

I was somewhat startled to find this book described as marking the end of an epoch in the Theory of Games, because that theory—with capital letters—deals with much more general and more abstract aspects of games than a book on probabilities in Bridge. But what prompts me to these notes is a feeling that students of the history of mathematics might be interested in a few facts that might get distorted in the course of time.

The Theory of Games did not start " with a problem posed a few years ago at the poker tables of Princeton ". It is true that Professor Braithwaite finished his delightful inaugural lecture on " Theory of Games as a tool for the Moral Philosopher " by speaking of a " radiation from a source—theory of games of strategy—whose prototype was kindled round the poker tables of Princeton ", but this was, of course, rhetorical. He also reminded his audience of the fact that " Emile Borel in 1921 thought of some of its basic ideas but he was held up by inability to establish the mathematically fundamental theorem ". (I am glad that Mr. Brookes alluded to Borel's contribution.) The latter theorem was established by John von Neumann. Now it is a moot point whether a theory starts with the conjecture of a theorem or by its proof. At any rate, the proof was communicated by von Neumann in a talk given to the Goettingen Mathematical Society in 1926 (published in Mathematische Annalen 100, in 1928), and there was no connection, at that time, with the poker tables of Princeton.

After this insistence on historical truth I think I ought to add that, since about 1944, it was indeed a galaxy of Princeton scholars who have made the Theory of Games what it is now: an edifice based on algebra, geometry, set theory, topology, economics, psychology and sociology, but by no means finished as yet.

Yours, etc., S. Vajda

129 Great Tattenhams, Epsom, Surrey

THE MATHEMATICAL TRIPOS

Important changes have recently been made in the lecture schedules for both Part I and Part II of the Cambridge Mathematical Tripos. These changes will affect both examinations for the first time in 1957. Perhaps some of your readers may be interested to hear of the needs which the new syllabuses are intended to meet.

The changes in Part I are the more important. Experience has revealed a number of defects in the Part I courses, when considered as preparation for the rest of the Mathematical Tripos. It was also felt that the courses could be made more useful for those, at present roughly 30% of all who take Part I, who then proceed to some other Tripos, such as the Natural Sciences or Economics. Especially, it was felt desirable to cater for the potential physicist who wished to proceed directly from Part I of the Mathematical Tripos to Physics in Part II of the Natural Sciences Tripos. To meet the needs of these different classes, a thorough revision and rearrangement of the schedules has been made. In addition as an alternative to a one-term course on conics, new courses on numerical analysis and statistics have been introduced, equivalent together to a single one term course. It was thought that these new courses would be more useful for example, to those intending to go on to economics or natural sciences, and would also prove attractive to many others.

In making these changes the Faculty have had to bear in mind also the difficulties of those who, when they come up, are uncertain whether to take Part I or the Preliminary Examination for Part II (the Mays) in their first year; or who, while taking Part I, wish to make a start with work for the Preliminary. The new arrangement of material for Part I brings it more into line with that for the Preliminary year. It is felt that too many mathematical students when they come up go on directly to do the Preliminary to Part II, and it is hoped that more will be encouraged to start with Part I.

The combination of differential equations with algebra in one course was always awkward. The natural affinity of the Part I algebra is with the geometry which exemplifies its methods. Algebra and geometry are now to be treated together in two terms. The first of these will include an introduction to matrices, and should pave the way to the linear algebra course for Part II, which is sometimes found difficult; the second of the two terms is devoted to the course on conics. The two terms on analysis have been somewhat lightened. In particular it seemed undesirable to combine in one course topics like partial differentiation, which can be treated with reasonable rigour, with others, such as multiple integrals, where only an intuitive treatment is suitable at the Part I stage. Consequently, multiple integrals are now to be dealt with, along with differential equations, vectors and solid geometry, in the first term course on " mathematical methods ". Other less important changes have also been made. For example, statics, which is felt to be a dull introduction to University mechanics, has been banished from the first term of the mechanics course. The Electromagnetism course has been altered so as to place more emphasis on the physical understanding of the subject.

The Part II courses extend, as before, over two years and the examination is on the whole of the work of those two years; but a number of new courses have now been added to those previously given in the second year. These new courses, on such subjects as atomic theory, topology, and a more realistic treatment of fluids (among others) are intended to give those who do not proceed to Part III the opportunity of a broader mathematical education, and those who do the chance of a somewhat earlier start on the line of their special interests. There will now be altogether thirteen courses for the second year of Part II work. But only two of these, the courses on abstract algebra and fluid mechanics, remain compulsory, like all those of the Preliminary

year, for the purposes of the Part II examination. Of the remaining eleven courses in the second year, candidates are expected to answer questions on four or five, at least, depending on the number of lectures involved. Thus the new scheme for Part II allows for a considerable freedom of choice and a certain degree of specialization which did not previously exist.

P. HALL

REPORTS ON COLLOQUIA

1. *Notes on a conference held at Oxford between schoolteachers and representatives from Universities and from industry*, 8–18 *April* 1957.

The following quotations from the document which was issued inviting schoolteachers to attend will give an idea of the objects of the conference.

" Scientific manpower shortage is already a grave national problem." " A situation already grave will in 20 years' time have become critical unless effective action is taken now and urgently." " This conference will give teachers an opportunity of exploring one aspect of education namely the teaching of mathematics. This is the main single ingredient of a scientific education at school." " At the conference teachers of mathematics will be able to meet industrial representatives and other users of mathematics and to discuss with them the teaching of mathematics in the light of national needs." " University lecturers will deal in non-specialist lectures with recent advances in mathematics."

Over 400 teachers applied to come, which indicates clearly that the importance of the conference was fully understood, particularly as many felt before it that 10 days was too large a proportion of the holidays. About 60 were selected—mainly those with good academic qualifications and considerable experience in the teaching of mathematics.

In most of us there was a rude awakening. We found ourselves almost totally ignorant of modern trends in mathematics and of its vital importance to industry, and very soon realized that 10 days at full pressure was none too long after all. There was a general feeling of stimulation and it is fair to say that even if there will be few radical changes in the syllabus taught, our actual teaching will be very much enlivened as a result of attending.

We were offered a choice of two series of lectures out of five. These were as follows :

Professor G. Temple	" Types of Fluid Flow "
Mr J. M. Hammersley	" Monte Carlo Methods "
Mr D. G. Kendall	" The Mathematical Study of Random Phenomena "
Mr J. G. Mauldon	" The Theory of Games "
Mr G. E. H. Reuter	" Non-Linear Oscillations "

We were given the opportunity of hearing each lecturer once before making a decision. It says much for their quality that the choice was an extremely difficult one and that once made no one was heard to complain of having chosen unwisely. Most of us had our heads above water for at least part of the time, and if we were sometimes battered out of understanding by the impact of shock waves or of random particles, we very soon recovered and were ready for further treatment. There is no doubt that these lectures were all much appreciated and very stimulating.

The representatives of industry gave us a very varied selection of talks. On the one hand we were given essentially mathematical lectures, of which two stand out in my memory. In the first we were invited to travel at super-

sonic speed both forwards and backwards with a very small interval of time between the two, and in the second we were shown a mastery of matrix technique in the solution of the problem of finding the current in each branch of a general plane electric circuit. On the other hand we were shown the various ways in which mathematicians were used in industry, and it was comforting to realize that there are opportunities for all grades from the University graduate with research experience to the child with no more than " O " Level. We were also shown the various methods used by the larger firms for ensuring that the maximum possible numbers were given those technical qualifications suited to their abilities, and we realized that very few trainees in industry were failing, through lack of opportunity, to reach their optimum level. In addition we were given an insight into the methods used in the United States for bringing on mathematicians and scientists and in particular into the amount of help given by the Federal Government for this purpose.

The standard of these talks was extremely high and there was a general feeling that all the speakers would be a most welcome addition to the teaching profession whether on account of their obvious enthusiasm or of their excellent lecturing technique.

In addition to these two main topics there was a series of discussions. The most illuminating was introduced by Dr F. W. Land of the Department of Education, University of Liverpool, who has conducted an enquiry into the numbers of graduates in University Departments of Education. These showed, particularly among women, the very serious shortage of teachers of mathematics and physics, especially those with first class or good second class qualifications. Suggestions for improving the situation were in general aimed at increasing the numbers rather than the quality of teachers and ranged from the introduction of a diploma of mathematics to persuading geography graduates to learn some mathematics during their year's training and so qualify to teach mathematics to the lower Forms. Television was suggested and although it was dismissed summarily by many of those present it is a possible field which should be looked into carefully before it is discarded as impracticable.

The work of technical colleges and colleges of technology was discussed. To many of us this was most illuminating and its importance can hardly be over-emphasized. These colleges are naturally in tune with the requirements of industry to a very much greater extent than the schools and the work in mathematics, though often of a very advanced nature, has in most cases immediate practical application.

Other subjects of discussion were : The use of girls of mathematical ability in industry in which the tendency to put them on to the more dull routine jobs was deplored, even though they were considered to be far more proficient at them than their male counterparts : the importance of closer liaison between schools, Universities, technical schools and industry, in which the importance of the most careful choice of representatives of industry to visit schools was emphasized : the allotment of periods to the various subjects for the mathematics and science specialist in which the importance of a breadth of outlook—stressed by many representatives of industry—led to the realization that a considerable proportion of the time-table—at least a third—should be allotted to non-specialist subjects : and many others.

Over the whole conference loomed the digital computer now being used to an ever increasing degree for the solution of problems. The resulting change in mathematical technique was brought clearly home, as was the fact that problems which before its advent were considered too difficult to solve, except in a very empirical way, were now relatively simple. Among many examples given, one of the most interesting was " linear programming " developed in

the first place in the United States and now used widely by many industries in which efficient collection, division and distribution of commodities was essential. For such problems the ability of the computer to deal rapidly with large numbers of linear equations and inequalities has enabled many great advances to be made.

The conclusions to be drawn are many. We now know the requirements of industry in their engineers and research workers. These are briefly a liveliness of mind and a breadth of vision combined with a willingness and ability to continue to learn. As industry is prepared to give the necessary vocational training the task of the schools is to give the boys or girls a good general mathematical and scientific background as well as a keen interest in other subjects and activities. This does not mean that no effort should be made in the schools to give the pupils an insight into the sort of problem they will have to face. An appreciation of statistical ideas, the approximate nature of nearly all the application of mathematics and above all the ability to formulate a mathematical problem out of diverse physical facts can all be taught to some degree to pupils at all stages. In addition among the many subjects suggested as suitable for teaching to the more advanced students the following seem to be the most fruitful : The approximate solution of differential equations and of integrals, iterative methods and matrices.

As a result of the conference there should be better relations between schools, Universities and industry and a better understanding of their several problems. Subsequent conferences to which it is intended science teachers should also be invited should enable even closer ties to be made.

Many people were responsible for the success of the conference. Our thanks are due to the President and Fellows of Trinity College for their hospitality, to those industrial firms who contributed financially and enabled us to attend for a very modest fee, to the University lecturers and industrial representatives, to the Oxford University Delegacy for Extra-Mural Studies and to many others. Above all we are deeply indebted to Mr Hammersley who was the prime instigator and who nursed the conference through many difficulties from its earliest beginnings. He was largely responsible for its undoubted success and all those present must be aware of what they owe to him.

This report is of necessity brief and the writer apologizes for any errors or omissions. Fortunately a full report is being published by *The Times* and is now available.

Wellington College M. D. PARKES

2. *Colloquium on History of Mathematics*

The third international colloquium on history of mathematics at the Mathematical Research Institute of Oberwolfach (Black Forest) was held on April 11–13 1957 under the Chairmanship of Prof. J. E. Hofmann (of Ichenhausen), who designed it to commemorate the birth of Euler (15. IV. 1707). He contributed in his closing address a psychological assessment of Euler's mode of thought, suggesting as its most typical characteristic a phenomenal penetration into masses of seemingly unconnected formulae, enabling him to combine and systematise them with a truly amazing power of imagination. This is already manifest in his attack on various special problems in summation and interpolation, started as a kind of sport in rivalry with his friends Bernoulli and Goldbach, at a time when he had access to practically no mathematical literature at St Petersburg. He built up general methods out of the many different *ad hoc* artifices employed, and in the very process of setting these methods on a strict and rigorous footing, laid the foundation of a new chapter in mathematical knowledge.

The following were the main points of interest from other speakers.

Problems in Euler's *Introduction to Algebra*, completed in 1767, can be

traced back to similar Babylonian arithmetic problems. Euler began dictating this work immediately after the final loss of his sight and probably incorporated much material dating from his Basle schooldays. (Dr K. Vogel, of Munich).

Euler's criticisms of Lagrange's and d'Alembert's theorems in two letters to Lagrange contribute materially to a clarification of the subject. A formal simplification of Euler's theorem by proceeding to the δ-method is discussed together with treatments of the problem of oscillating strings. (Prof. P. Funk, of Vienna).

The two most ancient (fragmentary) papyrus-ephemerides known to us, A.D. 348–9 and A.D. 467 respectively, are both calculated from the tables of Theon of Alexandria, not however quite without error, so that the task of correction and reconstitution is heavy. (Prof. J. J. Burkhard of Zürich, joint work with Prof. B. L. van der Waerden).

Three recent papers by Stamatis, of Athens (Akad.-Ber. 30, 1955 and 31, 1956), are linked with the now nearly completed modern Greek edition of Euclid. Theodorus' proof of the irrationality of square roots is reconstructed from Plato's Theaetetus (147 d. 3–6). Attempts in antiquity to explain why the individual proofs break off at $\sqrt{17}$ are analysed. And there is a discussion of Euclid's peculiar method of proof for the equality of the ratios of the areas to the squared radii for two given circles. (Prof. S. Heller of Schleswig).

The publication of Kepler's Collected Works has now reached Vol. 18, the 6th and last volume of Letters. Incidental to the preparation involved, indications by Schickhardt for the construction of a calculating machine were discovered. Prof. B. v. Freytag-Löringhoff of Tübingen undertook the reconstruction. The machine, though not wholly automatic, is usable, for sums, products, quotients and square roots. (Prof. Hammer of Stuttgart).

J. Gregory's *Geometria pars universalis* of 1668 provides our first connected account of the infinitesimal methods of his day. A recurrence formula for the area of a circular, elliptic or hyperbolic sector, linking arithmetic and geometric means, is given in his *Vera quadratura* of 1667. (Dr Chr. Scriba of Giessen).

A discussion on the changes in the concept of mathematical rigour since Euler's day followed a lecture by Dr N. Stuloff of Mainz on the subject, which brought out the fundamentals very clearly.

[From the report by Prof. J. E. Hofmann translated by Dr R. C. H. Tanner.]

3. *Colloquium on Number Theory*

From March 11th–16th a conference on number-theory was conducted at the Oberwolfach Institute by Professors Hoheisel, Rohrbach and Schneider.

The opening lecture was given by Prof. Volk who spoke with fine human detail about Lindemann and his proof of the transcendence of π. Other speakers were Prof. Schneider (Mahler's S-numbers), Dr Cassels and Prof. Kuipers (different aspects of Diophantine Approximation), Prof. Keller (Hajos's theorem), Dr Lekkerkerker (Mahler's compound bodies), Dr Richert (Riesz means of numbertheoretic functions and Turan's work on the Lindelöf conjecture), Herr Ehlich (Selberg's formula for quadratic forms), Prof. Kanold (Numbertheoretic functions), Mr Birch (Manin's elementary proof of the Riemann Conjecture for functionfields of genus 1) and Dr Herrmann, who spoke on his work on modular forms of the second degree.

For the later part of the conference we were joined by Profs. Rohrbach and Müller, Dr E. Härtter and Herr B. Müller; and attention shifted to additive number theory. Dr Wirsing gave two talks, in the second of which he proved this extension of a theorem of Romanov: if $f(x)$ is any polynomial with integer coefficients and p, q denote primes, then the numbers of the type $p+f(q)$ have positive asymptotic density. Prof. Stöhr gave two elegant talks

on " straight " additive numbertheory ; Dr Härtter spoke on minimal bases and then Dr Kasch and Prof. Rohrbach discussed possible generalizations of the theorems of Schnirelmann and Mann to integer lattices in spaces of several dimensions. The last talk was by Prof. Schneider, who gave an account of Roths work on rational approximations to algebraic *i* irrationals.

We left Oberwolfach on Sunday morning after an instructive and enjoyable week spent in beautiful and congenial surroundings.

B. J. Birch
J. W. S. Cassels

4. *Colloquium on Theory of Functions of several complex variables*

At the invitation of the Mathematical Research Institute at Oberwolfach (Director Prof. W. Süss of Freiburg-in-Breisgau), a colloquium, international in character, was held on March 19–22 on Theory of Functions of several complex variables, under Professors H. Behnke of Münster and K. Stein of Munich. The 18 papers read at the meeting covered the most recent work and demonstrated the importance of the theory of sheaves and fibre spaces for complex analysis. Discussion was as lively as always in the congenial atmosphere of the Lorenzenhof, particularly on the connection between complex functiontheory and the theory of real-analytic manifolds and sets. Prof. H. Cartan of Paris (" Sur les ensembles analytiques réels ") showed how much more complicated it is to set up basic theorems for real-analytic sets than for complex-analytic. Thus there is no unrestricted validity for the theorem on unique decomposition into irreducible components, nor for the identity theorem. For coherent real-analytic sets, however, a satisfactory theory can be set up. Dr B. Malgrange (" Sur les variétés analytiques réelles ") proved that a real-analytic manifold X^n is capable of being analytically imbedded in a real number-space R^n if and only if X^n is a real-analytic Riemannian manifold. Another necessary and sufficient condition is that the analogues of H. Cartan's theorems A and B for coherent complex-analytic sheaves remain valid.

Further summary. Dr R. Remmert of Münster (" Reduktion komplexer Räume ") : Reducibility of functiontheory in an arbitrary prescribed complex space to functiontheory in one that is holomorphically complete. A sufficient condition is that the prescribed space be holomorphically convex.

Prof. K. Stein of Munich (" Eigentliche holomorphe Abbildungen ") : Theorems of classical mapping theory generalised to strictly holomorphic mappings. Connection with the theory of analytic decomposition. Every holomorphic superimposed mapping of a simplex in C^n is linear, and, in particular, an automorphism. A finite number only of such mappings exist in general.

Prof. W. Thimm of Bonn (Über die Singularitäten meromorpher Abbildungen ") : Image sets of points of indetermination in meromorphic mappings.

Prof. F. Sommer of Münster (" Lokale Probleme der komplexen Differentialgeometrie ") : Differential form of conditions for real-analytic decompositions of complex manifolds to be locally complex-analytic decompositions.

Prof. H. Tornehave of Kopenhagen (" Fastperiodische Funktionen und Funktionentheorie mehrerer Veränderlichen ") : Connection between the theory of almost-periodic functions and that of functions of several complex variables.

Prof. P. Dolbeault of Paris (" Formes différentielles méromorphes localement exactes ") : The dimension of certain cohomology groups evaluated, to generalise theorems of Kodaira and Spencer.

Prof. P. Lelong of Lille (" Formes différentielles sur les ensembles analytiques) : Theory of differential forms on complex-analytic sets with singularities. Stokes' theorem.

Prof. N. H. Kuiper of Wageningen (" Kurze und isometrische Einbettungen "): C^1-imbeddings of C^1-manifolds.

Dr H. Röhrl of Munich (" Analytische Überlagerungen komplexer Räume "): Existence-criteria, in fibre-space phraseology, for algebroid functions with prescribed branchpoint behaviour.

Dr H. Grauert of Mainz (" Das Levische Problem für Gebiete über Steinsche Mannigfaltigkeiten ") : A basic theorem of Oka extended to Stein manifolds. A pseudo-convex Riemannian domain devoid of branchpoints over a Stein manifold is itself a Stein manifold.

Dr H. G. Tillmann of Mainz (" Analytische Funktionale ") : By introducing a suitable topology it is possible to set up a satisfactory theory of analytic functionals. The basic identity theorem results from Runge's general approximation theorem.

Dr J. Frenkel of Strassburg (" Sur le premier théoreme de Cousin dans certains domaines de C^n ") : A general class of domains, other than domains of holomorphy in C^n, for which the cohomology group $H^1(G, \theta)$ vanishes, and in which therefore every Cousin-I-distribution has a solution.

Dr R. J. Wille of Delft (" Starre topologische Räume ") : General process for the generation of rigid topological spaces.

Dr E. Schieferdecker of Munich and Münster (" Funktionentheorie in Banachschen Räumen ") : The group of analytic automorphisms of the hypersphere in special Banach spaces.

Dr W. Stoll of Tübingen (" Über den Graphen meromorpher Abbildungen ") : In significant cases, the graph of a meromorphic transformation is always an analytic set.

Dr H. Griesel of Münster (" Überkonvergenz in der Funktionentheorie mehrerer Veränderlicher ") : A number of classical theorems of Ostrowski extended to several complex variables.

[From the report by Prof. H. Behnke translated by Dr R. C. H. Tanner.]

PROBLEM

The linear cubic recurrence $u_{n+3}=2u_{n+2}-2u_{n+1}+u_n$ with $u_0=0$ has $u_k=0$ whenever k is a multiple of 6. A recurrence may have an infinite number of zeros when its companion polynomial (here x^3-2x^2+2x-1) vanishes for values which are roots of unity.

$u_{n+3}=u_{n+2}-u_{n+1}+\frac{1}{2}u_n$ with $u_0=u_1=0$ has $u_k=0$ when $k=0$, 1, 4, 6, 13 and 52. Can anyone produce a linear cubic recurrence with a finite number of zeros greater than six?

Singleton Lodge, Blackpool R. C. LYNESS

BOOKS WANTED

Ganguli : *Theory of Plane Curves*—Vol. I, 3rd ed. ; Vol. II, 2nd ed.

Hilton : *Groups of Finite Order.*

Lewent : *Conformal Representation.*

Muir and Metyler : *Determinants.*

Oakes : *Tables of Reciprocals.*

Watson : *Complex Integration and Cauchy's Theorem*—No. 15 in Cambridge Tracts.

Journal of Indian Mathematical Society—1909 to 1933.

Mathematical Questions from the Educational Times—1912 to 1918 both inclusive.

Anyone wishing to dispose of these is welcome to write to Clifford Marburger, Denver, Pennsylvania, U.S.A. Payment in advance of shipment.

2703. A curious result.

Casting about, trying to construct a dynamical example to illustrate invariant processes in vector and tensor algebra, I obtained the following rather curious result.

A body turns about a fixed point O under the action of no forces. Its angular velocity is $\boldsymbol{\omega}$ and the inertia tensor at O is $\mathbf{J}$ and the tensor of quadratic moments at O is $\mathbf{K}$. Then the rate of change of the vector $\mathbf{JK}\boldsymbol{\omega}\,(=\mathbf{KJ}\boldsymbol{\omega})$ relative to a frame moving with angular velocity $2\boldsymbol{\omega}$ is zero.

Further
$$\mathbf{H}\,.\,\mathbf{JH} - \omega^2 s_3 = \text{const.}$$
$$(\mathbf{JH})^2 - \omega^2 s_1 s_3 = \text{const.}$$

where $\mathbf{H}$ is the angular momentum at O, and s_1, s_2 and s_3 are the scalar invariants of the inertia dyadic, and with the usual notation

$$s_1 = A+B+C, \quad s_2 = BC-F^2+CA-G^2+AB-H^2, \quad s_3 = \text{det. } \mathbf{J}.$$

Moreover, $\mathbf{J}+\mathbf{K} = \frac{1}{2}s_1\mathbf{I}$ where $\mathbf{I}$ is the idemtensor.

The equation of motion is $\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{H}\wedge\boldsymbol{\omega}$.

Therefore
$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \mathbf{J}^{-1}(\mathbf{H}\wedge\boldsymbol{\omega})\\ &= (\mathbf{JH}\wedge\mathbf{J}\boldsymbol{\omega})/s_3 \quad \text{since } \mathbf{J} \text{ is symmetric.}\\ &= (\mathbf{JH}\wedge\mathbf{H})/s_3\end{aligned}$$

Therefore
$$\boldsymbol{\omega}\,.\,\dot{\boldsymbol{\omega}} = \frac{1}{2}\frac{d}{dt}\omega^2 = [\mathbf{JH}, \mathbf{H}, \boldsymbol{\omega}]/s_3 \quad \text{.........................(1)}$$

Also
$$\frac{d}{dt}\mathbf{JH} = \boldsymbol{\omega}\wedge\mathbf{JH} - \mathbf{J}(\boldsymbol{\omega}\wedge\mathbf{H}) \text{ since } \mathbf{H} \text{ is const.}$$

But $\mathbf{J}$ satisfies its characteristic equation $\mathbf{J}^3 - s_1\mathbf{J}^2 + s_2\mathbf{J} - s_3 = 0$ and therefore

$$s_3\boldsymbol{\omega} = s_2\mathbf{H} - s_1\mathbf{JH} + \mathbf{J}^2\mathbf{H}$$

and it follows that
$$\begin{aligned}\mathbf{J}(\boldsymbol{\omega}\wedge\mathbf{H}) &= (-s_1\mathbf{JH} + \mathbf{J}^2\mathbf{H})\wedge\mathbf{H}/s_3\\ &= (-s_1\mathbf{H} + \mathbf{JH})\wedge\boldsymbol{\omega}\end{aligned}$$

Thus
$$\frac{d}{dt}\mathbf{JH} = \boldsymbol{\omega}\wedge(2\mathbf{JH} - s_1\mathbf{H})$$
$$\frac{d}{dt}\left(\mathbf{JH} - \frac{s_1}{2}\mathbf{H}\right) = 2\boldsymbol{\omega}\wedge\left(\mathbf{JH} - \frac{s_1}{2}\mathbf{H}\right)$$

since s_1 is constant throughout the motion. This proves the first part.

Forming the scalar product with $\mathbf{H}$, we get :

$$\mathbf{H}\,.\,\frac{d}{dt}\mathbf{JH} = 2[\mathbf{H}, \boldsymbol{\omega}, \mathbf{JH}] = 2s_3\,\boldsymbol{\omega}\,.\,\dot{\boldsymbol{\omega}} \quad \text{by (1).}$$

Because $\mathbf{H}$ is constant, we get by integration

$$\mathbf{H}\,.\,\mathbf{JH} = s_3\omega^2 + \text{const.} \quad \text{..............................(2)}$$

Similarly
$$\begin{aligned}\mathbf{JH}\,.\,\frac{d}{dt}\mathbf{JH} &= -s_1[\boldsymbol{\omega}, \mathbf{H}, \mathbf{JH}]\\ &= s_1 s_3\,\boldsymbol{\omega}\,.\,\dot{\boldsymbol{\omega}}.\end{aligned}$$

Thus
$$(\mathbf{JH})^2 = s_1 s_3\,\omega^2 + \text{const.} \quad \text{..........................(3)}$$

In terms of H and the kinetic energy T, the equations (2) and (3) are equivalent to

$$s_1\mathrm{H}^2 - s_2 \,.\, 2T = \text{const.}$$
$$(s_1{}^2 - s_2)\mathrm{H}^2 - (s_1 s_2 - s_3)2T = \text{const.}$$

E. V. WHITFIELD

2704. A note on convergence (Note 2560).

The analysis given to prove that $n^r x^n \to 0$ as $n \to \infty$ (with $0 < x < 1$ and r any given integer) is no doubt much more elegant, but it is far easier for teaching purposes to use logarithms. For if $\log x = -k(k > 0)$, then

$$\log (n^r x^n) = r \log n - nk = -n(k - r \,.\, \log n/n),$$

which $\to -\infty$ on the assumption that $\log n/n \to 0$. From this it follows that $n^r x^n \to 0$.

The assumption that $\log n/n \to 0$ is one which the pupil will accept on the basis of his experience with logs to the base 10, by comparing for example the series 10, 10^2, 10^3, 10^4, ... for n with the series 1, 2, 3, 4, ... for $\log n$. A formal proof can be given later when the general theory of the logarithmic function has been developed.

C. O. TUCKEY.

2705. Length of a perpendicular (Note 2575).

If P is (x_1, y_1) and $Q(x_2, y_2)$ is foot of the perpendicular, then since PQ is perpendicular to $ax + by + c = 0$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$$

or, $$b(x_2 - x_1) - a(y_2 - y_1) = 0 \qquad \text{(i)}$$

also since (x_2, y_2) is on the line,

$$\therefore\ ax_2 + by_2 = -c$$

or, $$a(x_2 - x_1) + b(y_2 - y_1) = -(c + ax_1 + by_1) \qquad \text{(ii)}$$

Squaring and adding (i) and (ii)

$$(a^2 + b^2)d^2 = (ax_1 + by_1 + c)^2.$$

V. G. SEDERMAN

2706. Two theorems deduced from the theorems of Ceva and Carnot.

In two notes in earlier numbers of the Gazette, Mr. E. J. Hopkins* and Mr. M. P. Drazin† each gave a proof of a certain theorem on concurrency in a triangle. This theorem may be regarded as a generalisation or extension of the theorem on the concurrency property of the bisectors of the angles of a triangle. Is there perhaps a theorem which may be regarded as a similar generalisation of Ceva's theorem? The following is presented as such, and can be very easily deduced from Ceva's theorem :

Theorem 1 (*a*). Let D and D', E and E', F and F' be pairs of points on the sides BC, CA and AB respectively of the triangle ABC. Let BE and CF' intersect in X, CF and AD' in Y, and AD and BE' in Z. Then the necessary and sufficient condition for AX, BY and CZ to be concurrent is

* Vol. 34 (1950), note 2144, pp. 129–133.
† Vol. 37 (1953), note 2328, pp. 55–57.

$$\left(\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}\right)\left(\frac{AE'}{E'C}\cdot\frac{CD'}{D'B}\cdot\frac{BF'}{F'A}\right)=1 \quad \text{.....................(1)}$$

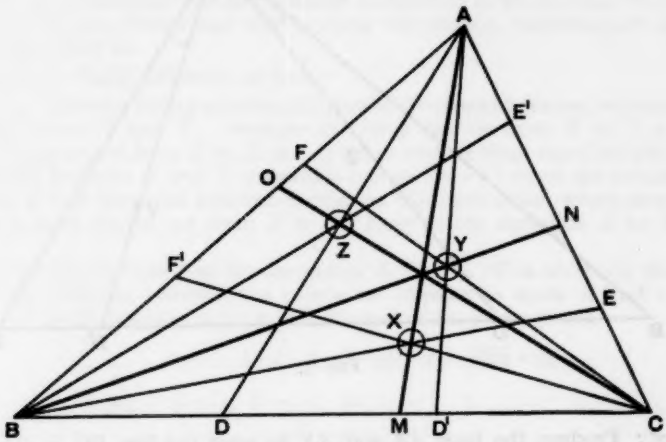


FIG. 1.

Proof: Let AX meet BC in M, BY meet CA in N, and CZ meet AB in O. Then, by Ceva's theorem

$$\frac{AE}{EC}\cdot\frac{BF'}{F'A}\cdot\frac{CM}{MB}=1 \quad \text{..............................(2)}$$

$$\frac{BF}{FA}\cdot\frac{CD'}{D'B}\cdot\frac{AN}{NC}=1 \quad \text{..............................(3)}$$

$$\frac{CD}{DB}\cdot\frac{AE'}{E'C}\cdot\frac{BO}{OA}=1 \quad \text{..............................(4)}$$

Therefore, from equations (2), (3) and (4)

$$\left(\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}\right)\left(\frac{AE'}{E'C}\cdot\frac{CD'}{D'B}\cdot\frac{BF'}{F'A}\right)\left(\frac{AN}{NC}\cdot\frac{CM}{MB}\cdot\frac{BO}{OA}\right)=1 \quad \text{.........(5)}$$

Thus, if equation (1) holds,

$$\frac{AN}{NC}\cdot\frac{CM}{MB}\cdot\frac{BO}{OA}=1 \quad \text{..............................(6)}$$

and AX, BY and CZ are concurrent. Conversely if AX, BY and CZ are concurrent then equation (6) holds and equation (1) follows as a consequence.

Q. E. D.

The theorem proved by Hopkins and Drazin can be derived from theorem (1*a*) as follows :

Theorem : If X, Y and Z are three points in the plane of a triangle ABC such that

$$Y\widehat{A}C = Z\widehat{A}B, \quad Z\widehat{B}A = X\widehat{B}C, \quad X\widehat{C}B = Y\widehat{C}A,$$

then the lines AX, BY and CZ concur.

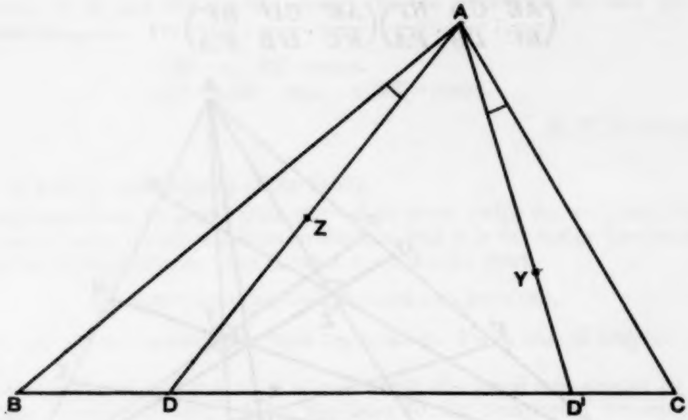


FIG. 2.

Proof: Produce the lines AZ and AY to meet the line BC in D and D'. Denote the equal angles by α.

Then, by the sine rule

$$\frac{CD'}{AC}=\frac{\sin\alpha}{\sin(\alpha+C)} \quad \text{and} \quad \frac{D'B}{AB}=\frac{\sin(A-\alpha)}{\sin(\alpha+C)}.$$

Therefore $$\frac{CD'}{D'B}=\frac{AC\sin\alpha}{AB\sin(A-\alpha)}. \qquad \text{.......(7)}$$

Similarly it may be shown that $$\frac{CD}{DB}=\frac{AC\sin(A-\alpha)}{AB\sin\alpha} \qquad \text{.......(8)}$$

Therefore $$\left(\frac{CD'}{D'B}\right)\left(\frac{CD}{DB}\right)=\left(\frac{AC}{AB}\right)^2. \qquad \text{.......(9)}$$

By a similar deduction $$\left(\frac{AE'}{E'C}\right)\left(\frac{AE}{EC}\right)=\left(\frac{AB}{BC}\right)^2$$

and $$\left(\frac{BF'}{F'A}\right)\left(\frac{BF}{FA}\right)=\left(\frac{BC}{AC}\right)^2.$$

Thus $$\left(\frac{AE'}{E'C}\cdot\frac{BF'}{F'A}\cdot\frac{CD'}{DB'}\right)\left(\frac{AE}{EC}\cdot\frac{BF}{FA}\cdot\frac{CD}{DB}\right)=\left(\frac{AC}{AB}\right)^2\left(\frac{AB}{BC}\right)^2\left(\frac{BC}{AC}\right)^2=1$$

[by Theorem 1 (*a*)]

and therefore AX, BY and CZ are concurrent.

Q.E.D

A second theorem may be deduced from Carnot's theorem and theorem 1 (*a*), as follows:

Theorem 1 (*b*): An alternative condition for AX, BY and CZ to be concurrent is that the points D, D', E, E', F, F' should all lie on a conic section.

This result follows directly from theorem 1 (*a*) and the theorem of Carnot in Projective Geometry.

R. M. WALKER

2707. Crossing the desert

A lorry when fully loaded can carry enough fuel to take it half-way across a desert. What is the minimum amount of fuel required to take it across the desert? It is assumed that any amount of fuel can be taken from the lorry at any point in the desert and this amount will remain undiminished till subsequently collected.

SOLUTION. $7\frac{2021}{3003}$ full loads of fuel.

PROOF. Consider in any solution the part of the desert between two successive storage points X and Y. Suppose the lorry travels from X to Y n times. Then it will travel from Y to X $(n-1)$ times and we shall say that the part of the desert between X and Y consumes fuel at $(2n-1)$ times the normal rate. Further if k is the total amount of fuel (in full load units) which during the whole journey starts out from X in the lorry in the direction X to Y, then $n \geqslant k$.

Now divide the desert up by the points $A, B, C \ldots G$ as shown in the figure where the distance between the points are marked in units of half a desert width. P is the starting point and Q the end of the journey and

$$a = 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{13} = \tfrac{2021}{3003} \times \tfrac{1}{15}$$

$$<a><\tfrac{1}{13}><\tfrac{1}{11}><\tfrac{1}{9}><\tfrac{1}{7}><\ldots\tfrac{1}{5}\ldots><\ldots\tfrac{1}{3}\ldots><\ldots\ldots1\ldots\ldots>$$

$P \quad G \quad F \quad E \quad D \quad C \quad B \quad A \quad Q$

Let $t(S)$ be the algebraic total amount of fuel (in full load units) which during the whole journey passes the point S (a general point) in the lorry measured positively in the direction S to Q. Let N be the normal rate of consumption of fuel. Then any part of the desert nearer to P than S must consume fuel at a rate $\geqslant [2t(S)-1]N$.

Certainly any part of AQ must consume at a rate $\geqslant N$ and hence $t(A) \geqslant 1$. Consequently $t(S) > 1$ if S lies nearer to P than A, and hence any part of BA must consume at a rate $\geqslant 3N$. Therefore

$$t(B) \geqslant \tfrac{1}{3} \times 3 + t(A) \geqslant 2.$$

Consequently $t(S) > 2$ if S lies nearer to P than B, and hence any part of CB must consume at a rate $\geqslant 5N$. Therefore

$$t(C) \geqslant \tfrac{1}{5} \times 5 + t(B) \geqslant 3.$$

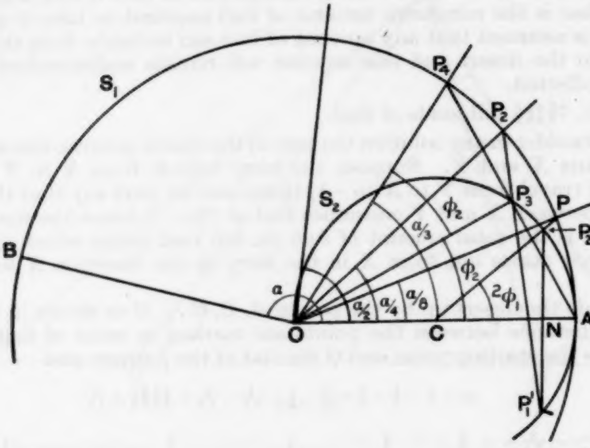
Continuing in this way we obtain $t(G) \geqslant 7$. Consequently $t(S) > 7$ if S lies between P and G, and hence any part of PG must consume at a rate $\geqslant 15N$. Therefore

$$t(P) \geqslant a \times 15 + t(G) \geqslant 7\tfrac{2021}{3003}$$

It is easy to see that this minimum can be attained. The simplest method is to use A, B, C, D, E, F and G as the only storage points and start out from P 7 times fully loaded, leaving $(1-2a)$ fuel units at G each time and returning to P. Start out from P the eighth time with the remaining fuel, $15a$ units, and thus arrive at G with a complete total of 7 units. We can now start out from G 7 times fully loaded, leaving $\frac{11}{13}$ fuel units at F on each of the first 6 occasions and returning to F. We thus arrive at F for the seventh time with a complete total of 6 units. Continuing in this way from F to E and E to D etc., we eventually arrive at A for the second time with $\frac{2}{3}$ units on the lorry and $\frac{1}{3}$ units already unloaded at A; just enough to fully load the lorry and take it to Q.

G. G. ALWAY

2708. Approximate trisection of an angle by ruler and compasses.



One of the three famous problems of antiquity is the trisection of a given angle by geometrical construction. Although this is incapable of exact solution by ruler and compasses alone, many methods of approximate trisection are known. Mr. Sydney Mather, a geometrical drawing master in Acton Technical Secondary School, has evolved an extremely simple geometrical construction yielding results of surprising accuracy. This is contained in the above figure and the following is an explanation of it.

Let the angle be AOB ($=\alpha$, say). Then the construction is to draw a circle S_1 centre O and any radius OA ($=a$) and a second circle S_2 of radius OC ($=CA=\frac{1}{2}OA=\frac{1}{2}a$), centre C, which thus touches S_1 at A. Bisect $\angle AOB$ three times successively by lines which thus make with OA the angles $\alpha/2$, $\alpha/4$, $\alpha/8$. Let the last two lines cut S_1 at P_2, P respectively and join CP cutting S_2 at P_1. Reflect P_1 in OA to P'_1 (on S_2) and join P'_1P_2 cutting S_2 at P_3. Join AP_3 cutting S_1 in P_4. Then OP_4 trisects $\angle AOB$ with a negative error commencing with zero for $\alpha=0$ and increasing to a maximum value of $0°\ 2'\ 11''$ for $\alpha=189°\ 33'\ 49''$ and then diminishing rather more rapidly to zero for $\alpha=249°\ 32'\ 0''$. After this there is a positive error increasing to its maximum value of $0°\ 29'$ for $\alpha=360°$, an angle which, incidentally, Euclid would never have recognised.

The degree of accuracy of the above construction can be gauged by using coordinate geometry which leads, according to the working of Dr A. J. Carr, to the following two cognate formulae, viz. in succession

$$\tan 2\phi_1 = 2\sin\tfrac{1}{8}\alpha/(2\cos\tfrac{1}{8}\alpha - 1) \quad \text{.......................................}(1)$$

$$\tan\tfrac{1}{2}x = \frac{2\cos(\phi_1+\alpha/12)-\cos(\alpha/6+\phi_1)-\cos(\alpha/6-\phi_1)}{2\sin(\phi_1+\alpha/12)+\sin(\alpha/6+\phi_1)+\sin(\alpha/6-\phi_1)} \quad \text{..........}(2)$$

where

$$\angle ACP = 2\phi_1, \qquad \angle AOP_4 = \alpha/3 - x.$$

Although it is not easy to eliminate ϕ_1 between (1) and (2), a few simple cases

serve to reveal that as α increases from zero to 360°, x increases to a maximum for $\alpha = \alpha_1$ (say), then diminishes to zero for $\alpha = \alpha_2$, afterwards becoming negative all the way from $\alpha = \alpha_2$ to $\alpha = 360°$.

To find this maximum we regard α as the independent variable and differentiate (1) and (2), using the first result in order to eliminate $d\phi_1/d\alpha$ from the second. Then, as usual, we put $dx/d\alpha$ zero. After some rather involved algebra it transpires that if $\cos \frac{1}{8}\alpha_1$ be c_1 then c_1 is a root of the algebraic quintic

$$64c^5 + 17c^4 - 144c^3 - 19c^2 + 79c + 1 = 0 \quad \text{......................(3)},$$

giving, by Horner's Method $c_1 = 0{\cdot}9156946$

$$\therefore \quad \tfrac{1}{8}\alpha_1 = 23° \; 41' \; 43{\cdot}63'', \quad \alpha_1 = 189° \; 33' \; 49{\cdot}04'',$$

on using Chambers's Seven Figure Mathematical Tables.

As α increases x now diminishes reaching zero when $\alpha = \alpha_2$, such that c_2 $(= \cos(\alpha_2/24))$ is a root of the quintic equation

$$c^5 + c^4 - c^3 - c^2 + 1/16 = 0.$$

Here Horner's Method yields $c_2 = 0{\cdot}9835802$, $\alpha_2 = 249° \; 32' \; 0''$.

Actually the decrease in x as α varies from α_1 to α_2 is seen to be much more rapid than the increase in x for $0 \leqslant \alpha \leqslant \alpha_1$. The rapidity continues until at $\alpha = 360°$ x is numerically equal to $29'$, its largest possible value.

Algebraically (3) has also the negative root c_3, where

$$\cos \tfrac{1}{8}\alpha_3 = c_3 = 0{\cdot}0126236 = \cos 90° \; 43' \; 23{\cdot}86''.$$

But then the value $\alpha_3 (= 725° \; 47' \; 11'')$, although producing an *algebraic* minimum for x, is utterly impossible geometrically.

The following numerical examples worked out by the aid of Seven Figure Tables applied to equations (1) and (2) in succession give some picture of the general trend of the angle x between the true and false trisectors.
(*a*) $\alpha = 60°$, $x = 16''$; (*b*) $\alpha = 90°$, $x = 32''$; (*c*) $\alpha = 120°$, $x = 1' \; 5{\cdot}7''$; (*d*) $\alpha = 180°$, $x = 2' \; 4\frac{1}{2}''$; (*e*) $\alpha = \alpha_1$, $x = 2' \; 11''$ (max.); (*f*) $\alpha = \alpha_2$, $x = 0{\cdot}016''$; (*g*) $\alpha = 360°$, $x = -29'$. Put more vividly, the true and false trisectors starting from 0 would be one inch apart after travelling in (*a*) a quarter of a mile, in (*e*), the maximum positive case, 50 yards, in (*g*), the maximum negative case, 11 feet and in (*f*) 211 miles!

S. Mather and A. J. Carr

2709. The quotient of two quadratics.

It is easy to obtain numerical examples of the quotient of two quadratics having rational zeros, poles and turning points.

For consider

$$f(x) = \frac{(x-a)^2}{(x-b)(x-c)} - \frac{(m-a)^2}{(m-b)(m-c)},$$

where a, b, c and m are rational.

(i) $f(x)$ is clearly expressible as the quotient of two quadratics with rational coefficients.

(ii) There are poles at $x = b$, c.

(iii) One of the zeros of $f(x)$ is at $x = m$, which is rational, so that the other zero must be rational also.

(iv) Thanks to the presence of the repeated factor $(x-a)^2$, one of the two zeros of $f'(x)$ is at $x = a$, which is rational, so that once again the second zero is rational.

Auckland University College. C. M. Segedin

2710. A propos de la note No. 2637.

Il est intéressant de noter aussi la *construction* du quatrième point commun M_4 à deux paraboles circonscrites à un triangle $M_1M_2M_3$ et dont les axes sont parallèles à deux droites rectangulaires données d, d'.

Les coniques du faisceau ponctuel déterminé par les deux paraboles marquent sur la droite impropre les couples de l'involution dont les points unis sont les points impropres de d, d' et dont un couple est donc formé par les points cycliques. Cette involution est projetée à partir d'un point propre quelconque suivant une involution équilatère dont les couples sont formés de droites également inclinées sur la direction de d.

Dès lors, M_4 se trouve à la fois sur le cercle circonscrit au triangle $M_1M_2M_3$ et sur les trois droites menées par M_1, M_2, M_3 et inclinées sur d comme le sont respectivement les droites M_2M_3, M_3M_1, M_1M_2.

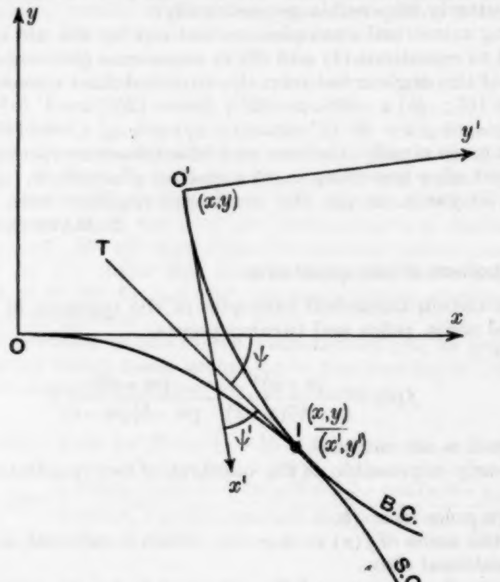
Les coordonnées de M_4 peuvent donc s'obtenir en résolvant le système linéaire des équations, qui s'écrivent immédiatement, de deux de ces droites.

Faculté polytechnique, Mons, Belgium. R. DEAUX

2711. Coincidence of the instantaneous centre and the acceleration centre in initial motion.

In problems on the initial motion in its plane of a lamina from rest under given non impulsive forces the initial instantaneous centre is also the point of no acceleration. This fact is very useful, for then initial accelerations relative to this point of the body, of the type $a\dot{\omega}_0$, are the true accelerations in space.

Although the result can often be inferred in particular cases it seems desirable to prove it generally. It will be sufficient for this purpose to consider the rolling without slipping, of the body centrode (B.C.) on the space centrode (S.C.). (The trivial case of initial motion without rotation is not considered.)



Take axes Ox, Oy with Ox tangent to the S.C. at O, and axes $O'x'$, $O'y'$ fixed in the body with $O'x'$ tangent to the B.C. at O', which at $t=0$ coincides with O.

When the instantaneous centre is at I the angle rolled through by body is $\theta = \psi + \psi'$, where ψ, ψ' are the inclinations of the common tangent to Ox, $O'x'$ respectively.

Now for I if $s = \text{arc } OI = \text{arc } O'I$

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = -\sin\psi$$

and
$$\frac{dx'}{ds} = \cos\psi', \quad \frac{dy'}{ds} = \sin\psi'.$$

$$\therefore \dot{x} = \dot{s}\cos\psi,\ \dot{y} = -\dot{s}\sin\psi,\ \dot{x}' = \dot{s}\cos\psi',\ \dot{y}' = \dot{s}\sin\psi'.$$

Differentiating again with respect to t and putting $t = 0$ we get

$$\ddot{x}_0 = \ddot{x}'_0 = \ddot{s}_0 \quad \text{.................................(1)}$$

$$\ddot{y}_0 = \ddot{y}'_0 = 0 \quad \text{.................................(2)}$$

Now if (X, Y) are the co-ordinates of O' in reference to Ox, Oy then putting

$$z = x + iy, \qquad Z = X + iY$$

$$z' = x' + iy',$$

we have in general position of I

$$z = Z + z'e^{-i\theta}$$

$$\therefore \dot{z} = \dot{Z} + \dot{z}'e^{-i\theta} + z'e^{-i\theta}. - i\dot{\theta}.$$

Differentiating again and putting $t = 0$ we get

$$\ddot{z}_0 = \ddot{Z}_0 + \ddot{z}'_0 \quad \text{.................................(3}$$

By (1), (2), (3) it follows that

$$\ddot{Z}_0 = 0.$$

Showing that the initial instantaneous centre considered as a point of the body has zero acceleration initially, which proves the theorem.

W. Davidson

2712. A proof of the formula for the arc-length of a curve.

In treatments of plane curves, the formula

$$s(t) = \int_a^t |f'(t)|\,dt \quad \text{.................................(1)}$$

for the arc length of a rectifiable curve is established. Here f is a continuous complex-valued function of a real parameter $t (a \leqslant t \leqslant b)$ usually assumed to have a continuous derivative f' in (a, b). It is proposed here to give a more elementary proof than is usually found in such treatments.

The arc length is defined as the upper bound (when it is finite) of the lengths of polygons "inscribed" to the curve. Supposing f given as continuous and with a continuous derivative in a subinterval (α, β), let $\alpha = t_0 < t_1 < t_2 \ldots < t_n = \beta$ be any subdivision of (α, β). The length of the inscribed polygon is then

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} f'(t)dt \right|$$

$$\leqslant \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f'(t)|\,dt$$

$$= \int_\alpha^\beta |f'(t)|\,dt. \quad \text{...........................(2)}$$

The right-hand side being independent of the particular subdivision, the lengths of the inscribed polygons are bounded above and hence the arc is rectifiable.

If the range (a, b) can be divided into a finite number of subintervals, in each of which f has the properties just assumed, the curve is called regular. We see that a regular curve is rectifiable, and that (2) still holds.

Taking the upper bound in (2) over all possible subdivisions, of a sub-interval (α, β) of the interval (a, b), we find

$$s(\beta) - s(\alpha) \leqslant \int_\alpha^\beta |f'(t)|\, dt,$$

from which it follows that s is continuous in (a, b). Combining this with the fact that $|f(\beta) - f(\alpha)| \leqslant s(\beta) - s(\alpha)$ which is immediate from the definition of s and means geometrically that the chord does not exceed the arc-length, we find

$$\left|\frac{f(t) - f(t_0)}{t - t_0}\right| \leqslant \frac{s(t) - s(t_0)}{t - t_0} \leqslant \frac{1}{t - t_0}\int_{t_0}^{t} |f'(t)|\, dt$$

for any two points t, t_0 of (a, b). Passing to the limit, we find that

$$s'(t_0) = |f'(t_0)|$$

at any point t_0 at which $|f'(t)|$ is continuous. Since this fails to happen at only a finite number of points, the desired formula (2) follows by integration.

W. F. Newns

2713. The mid-points of the three diagonals of a quadrilateral are collinear.

Let $M_1(x_1y_1)$, $M_2(x_2y_2)$, $M_3(x_3y_3)$, $M_4(x_4y_4)$, be a quadrilateral ; $M_5(x_5y_5)$ the intersection of M_1M_2 and M_4M_3, and $M_6(x_6y_6)$ the intersection of M_1M_4 and M_2M_3.

$$4\begin{vmatrix} \tfrac{1}{2}(x_1+x_3) & \tfrac{1}{2}(y_1+y_3) & 1 \\ \tfrac{1}{2}(x_2+x_4) & \tfrac{1}{2}(y_2+y_4) & 1 \\ \tfrac{1}{2}(x_5+x_6) & \tfrac{1}{2}(y_5+y_6) & 1 \end{vmatrix} =$$

$$= (x_1y_2 + x_2y_5 + x_5y_1 - x_5y_2 - x_2y_1 - x_1y_5) + (x_1y_4 + x_4y_6 + x_6y_1 - x_6y_4 - x_4y_1 - x_1y_6)$$
$$+ (x_6y_3 + x_3y_2 + x_2y_6 - x_2y_3 - x_3y_6 - x_6y_2) + (x_5y_3 + x_3y_4 + x_4y_5 - x_4y_3 - x_3y_5 - x_5y_4)$$

But each parenthesis is zero because $M_1M_2M_5$, $M_1M_4M_6$, $M_6M_3M_2$, $M_5M_3M_4$ are collinear. The result follows.

G. N. Vlahavas

2714. Proportional division.

Let AOB be a quadrant of a circle centre O and let C, D be the points of trisection of the quadrant arc. Take any point G on the middle third of the arc and produce OG to P making $OG = GP$. Let E, F be the feet of the perpendiculars from P to OB, OA and let EFG cut the quadrant again at H. Then H divides the arc AB in the same ratio in which G divides the arc CD.

Proof. Let J be the foot of the perpendicular from O to EF and let angle EOJ contain $\alpha°$ and GOJ contain $\beta°$, and therefore since angles GOJ, HOJ are equal and angles EOJ and FOG are equal,

$$\alpha = 90 - \alpha + \beta.$$

The point G divides the arc CD in the ratio $(\alpha - \beta - 30)/(\alpha - 30)$ and H divides the arc AB in the ratio $(90 - \alpha - \beta)/(\alpha + \beta)$ and

$$\frac{90 - (\alpha+\beta)}{(\alpha+\beta)} = \frac{180 - 3\alpha}{3\alpha - 90} = \frac{60 - \alpha}{\alpha - 30} = \frac{\alpha - \beta - 30}{\alpha - 30}.$$

Willesden, Holywood, Co. Down A. H. Finlay

2715. From scripts.

(i) Project the directrices to infinity. Therefore the ellipse becomes a circle, and the two foci become the centre of the circle.

(ii) $$(x^2+y^2-6ax+5a^2)(a^2p^4+4a^2p^2-6a^2p^2+5a^2)$$
$$=(ap^2x+2apy-3ax-3a^2p^2+5a^2)^2,$$
which is seen to be true on inspection of coefficients.

(iii) The curve is symmetrical about $y=\infty$.

(iv) " . . . and it is obvious that U, V, W are the centres of the escribed circles [the result whose proof was asked] and with time I could have proved it to be true."

E. A. MAXWELL

2716. Duplication formulae.

When in analysis the theory of the trigonometrical functions is based on the series for sin and cos, the proof that these functions sometimes vanish is usually carried out by substituting numerical values for the variable. This is done by Whittaker and Watson, Landau, Goodstein, to mention a few.

In the first section we show this is unnecessary. In the following sections we consider further consequences of the duplication formulae.

1. Consider a function $C(x)$, real-valued, defined for x positive and satisfying

$$C(2x)=2C^2(x)-1. \qquad \text{.................................(1)}$$

Examples of such functions are cos kx and cosh kx.

Suppose further $C(x)\leqslant 1$ and that $C(x)$ is always positive.

We prove then that $C(x)\equiv 1$.

For $\qquad C^2(2x)\leqslant C(2x)=2C^2(x)-1.$

Hence $\qquad 2(1-C^2(x))\leqslant 1-C^2(2x).$

$$1-C^2(x)\leqslant \tfrac{1}{2}[1-C^2(2x)]$$
$$\leqslant \tfrac{1}{4}[1-C^2(4x)]\ldots$$
$$\leqslant \frac{1}{2^n}[1-C^2(2^nx)]<\frac{1}{2^n}.$$

Let n tend to infinity. Thence

$$C^2(x)\equiv 1, \quad C(x)\equiv 1.$$

Cor. If $C(x)$ is sometimes less than unity, then it is sometimes non-positive. Hence if it is continuous, it sometimes vanishes.

2. The above argument plainly holds if we only assume (1) for one value $x=a>0$ with $C(a)>0$ and for all x of form 2^na, where n runs over all positive integers.

Let $C(x)$ be continuous, not constant, and never greater than unity. Hence it sometimes vanishes. Let ϖ be its least (positive) root, and suppose that $a<\varpi$.

We can find k, $0<k<\pi/2a$, so that $C(x)=\cos kx$ when $x=a$. Now assume (1) for $x=2^na$ where n runs over all positive *and negative* integers.

Extend the definition of $C(x)$ so that $C(-x)=C(x)$ when $-\varpi<x<0$ and $C(x+2\varpi)=-C(x)$ for all x.

If a/ϖ is of the form $m.2^n$ where m, n are integers, this is a postulate, since $C(x)$ will be defined in two ways for some x, but otherwise it is a true definition.

Then $C(x)$, cos kx agree for values of x of form $2^ma\pm 2^mr\varpi$, where m, n, r are integers.

3. Now drop the hypothesis of periodicity and assume (1) only for $a \,.\, 2^{-n}$ where $a<\varpi$ and n runs over all positive integers. Further, let $C(x)$ be continuous (and hence $C(0)=1$) and suppose it has a second derivative at $x=0$.

Then by L' Hôpital's theorem,

$$\lim_{x\to 0} \frac{C(x)-1}{x^2} = \lim_{x\to 0} \frac{C'(x)}{2x} = \tfrac{1}{2}C''(0).$$

Let k be the least positive integer such that $C(a)=\cos ka$. Then $C(x)=\cos kx$ for $x=2^{-n}a$.

Hence as $C''(0)$ exists, it equals the second derivative of $\cos kx$ at $x=0$, that is, $-k^2$.

Thus if we assume (1) also for $b \,.\, 2^{-n}$ where $b<\varpi$, and n runs over all positive integers and we take k_1 to be the least positive integer so that

$$C(b)=\cos k_1 b,$$

then $k=k_1$.

Finally, if (1) be assumed for all x and $C''(0)$ exists, then $C(x)=\cos kx$ for some k.

The hyperbolic case is similar.

4. We now start afresh and assume only that (1) holds in some interval to the right from $x=0$ and that $C(x)$ has a second derivative at $x=0$.

Lemma 1. Let $f(2x)=2f(x)$ for $x>0$. We do not assume $f(x)$ is everywhere continuous, but suppose it has a right-hand derivative at $x=0$.

Then
$$\frac{f(x)}{x} = \frac{f(x/m)}{x/m}$$
where $m=2^n$ and n is any positive integer.

Let m tend to infinity, then the right-hand side tends to $f'(0)=k$, say. Hence $f(x)=kx$.

Lemma 2. Let $g(x)$ have a right-hand derivative at $x=0$, $g(0)\neq 0$, and $g(2x)=[g(x)]^2$ for $x>0$.

Hence $g(0)=1$. Suppose $g(x)$ not constant.

Now
$$\lim_{x\to 0} \frac{\log g(x)}{g(x)-1} = \lim_{y\to 1} \frac{\log y}{y-1} = 1\,;$$
then
$$\lim_{x\to 0} \frac{\log g(x)}{x} = \lim_{x\to 0} \frac{\log g(x)}{g(x)-1} \,.\, \frac{g(x)-1}{x} = g'(0) = c, \text{ say.}$$

Hence if $f(x)=\log g(x)$, then $f'(0)=c$ Also $f(2x)=2f(x)$.

Hence by Lemma 1, $f(x)=cx$, $g(x)=e^{cx}$.

Now let $C(x)$ satisfy (1) for all x. Consider the case when $C(x)\geqslant 1$, but is not constant. Suppose $C'(0)$ exists on the right, then

$$\frac{C(2x)-1}{2x} = \frac{C(x)-1}{x}\,[C(x)+1].$$

Let $x\to 0$. Then $C'(0)=2C'(0)$.

Hence $C'(0)=0$.

Define $S(x)=+\sqrt{[C^2(x)-1]}$, $x\geqslant 0$,

$$S(-x)=-S(x),$$

and
$$g(x)=C(x)+S(x).$$

Then $g(2x)=g(x)^2$.

We want to show that $g'(0)$ exists on the right. This will be done if we prove that

$$\lim_{x \to 0} S(x)/x \text{ exists}$$

or that $$\frac{S^2(x)}{x^2} = \frac{C(x) - 1}{x^2} \quad [C(x) + 1]$$

has a limit as x tends to 0 from above ; and this will be true if $C''(0)$ exists on the right, since then by L' Hôpital's theorem,

$$\lim_{x \to 0} \frac{C(x) - 1}{x^2} = \tfrac{1}{2} C''(0).$$

With this assumption, Lemma 2 gives

$$C(x) + S(x) = e^{cx}.$$

Hence $$C(x) - S(x) = e^{-cx},$$

$$C(x) = \tfrac{1}{2}(e^{cx} + e^{-cx}) = \cosh cx.$$

Auckland University College H. G. Forder

2717. On a duplication formula.

In the previous note Prof. Forder has shown that the only continuous, non-constant, real-valued, even solutions of

$$\phi(2x) = 2\phi^2(x) - 1 \quad \text{.................................(1)}$$

which are twice differentiable at $x=0$ are $\cos kx$ and $\cosh kx$. If we do not assume the existence of $\phi''(0)$ more interesting situations are possible.

1. Assume first that $\phi(x)$ is real, even, satisfies (1) and is continuous at $x=0$ with $\phi(0)=1$. Then $\phi'(0)$ exists and is zero.

There will be numbers M, δ such that $0 \leqslant \phi(x) \leqslant M$ for $0 \leqslant x \leqslant 2\delta$. Let $\cosh k\delta = M$, then in the interval $\delta \leqslant x \leqslant 2\delta$ we shall have,

$$\cos \frac{\pi x}{2\delta} \leqslant 0, \quad M \leqslant \cosh kx,$$

and so for all x. $$\cos \frac{\pi x}{2\delta} \leqslant \phi(x) \leqslant \cosh kx.$$

Hence $\phi'(0)$ exists and is zero.

We note that we can construct solutions of (1) of this type by taking any positive number δ, assigning arbitrary positive values for $\phi(x)$ in $\delta \leqslant x < 2\delta$ and requiring $\phi(x)$ to be positive for $0 \leqslant x < \delta$. No restrictions have to be put on the selected values to ensure the continuity and differentiability at $x=0$.

2. Assume next that $\phi(x)$ is defined for $x \geqslant 0$ and is

(i) real, and not constant,

(ii) bounded in each finite interval,

(iii) convex,

(iv) $\phi(0)=1$,

(v) $\phi(x)$ satisfies (1).

These functions form a much more restricted set than those considered in § 1. However, we find that solutions other than $\cosh kx$ are possible. We first show that for every non-constant solution $\phi(x) > 1$ for $x > 0$, and then the argument of § 1 shows that $\phi'(0)$ exists as a right-hand derivative and is zero.

First we note that (1) shows that $\phi(x) \geqslant -1$ for all x.

If there is a number a for which $0 < \phi(a) < 1$, then

$$1 + \phi(2a) = 2\,\phi^2(a) < 2\,\phi(a),$$

which is contrary to the convexity of $\phi(x)$.

If there is a number a for which $\phi(a)=0$, then $\phi(2a)=-1$, $\phi(4a)=1$, $\phi(8a)=1$, and the convexity is again contradicted.

If there is a number b for which $-1<\phi(b)<0$, unless one of the numbers $2^n b$ could be taken as a in one of the preceding cases, $\phi(2^n b)$ is negative for all positive integers n. This is impossible because when $\phi(x)$ and $\phi(2x)$ both lie between 0 and -1, the formula (1) requires $\phi(x)+\frac{1}{2}$ and $\phi(2x)+\frac{1}{2}$ to be both zero or to have opposite signs.

Now let P_0 be a point on the curve $y=\phi(x)$ and let its coordinates be $(\alpha, \cosh k\alpha)$. For each positive or negative integer n the point P_n with coordinates $(2^{-n}\alpha, \cosh 2^{-n}k\alpha)$ is also on the curve. For $\alpha<x<2\alpha$ the curve lies below its chord P_0P_{-1}. The corresponding property of the arcs P_nP_{n-1} for positive n imposes a further restriction on the arc P_0P_{-1}, although it does not force it right down to the curve $y=\cosh kx$.

If the point (x, y) is on the arc P_0P_{-1}, then $[2^{-n}x, \cosh(2^{-n}\cosh^{-1} y)]$ lies below the chord P_nP_{n-1}. If we write $Y=\cosh^{-1} y$, we find that for all n

$$\frac{\cosh 2^{-n}Y-\cosh 2^{-n}k\alpha}{x-\alpha}\leqslant\frac{\cosh 2^{1-n}k\alpha-\cosh 2^{-n}k\alpha}{\alpha}.$$

This gives

$$\sinh^2 2^{-n-1}Y-\sinh^2 2^{-n-1}k\alpha\leqslant\frac{x-\alpha}{\alpha}(\sinh^2 2^{-n}k\alpha-\sinh^2 2^{-n-1}k\alpha).$$

Multiplying through by 2^{2n+2} and letting $n\to+\infty$, we find

$$Y^2-k^2\alpha^2\leqslant\frac{x-\alpha}{\alpha}(3k^2\alpha^2).$$

So for $\alpha<x<2\alpha$ the points of the curve $y=\phi(x)$ must lie on or below the curve

$$y=\cosh k\sqrt{(3\alpha x-2\alpha^2)}. \qquad\text{.................................(2)}$$

This curve intersects $y=\cosh kx$ at $x=\alpha$ and at $x=2\alpha$.

If we now define a function $f(x)$ for $x\geqslant 0$ by

(i) $f(0)=1$,

(ii) $f(x)=\cosh k(3\alpha x-2\alpha^2)^{\frac{1}{2}}$ for $\alpha\leqslant x\leqslant 2\alpha$,

(iii) $f(x)$ satisfies $f(2x)=2f^2(x)-1$,

this function is convex, and so the assertion made at the beginning of § 2 has been established.

$$f'(x)=\frac{3k\alpha\sinh k(3\alpha x-2\alpha^2)^{\frac{1}{2}}}{2(3\alpha x-2\alpha^2)^{\frac{1}{2}}},$$

$$f''(x)=-\frac{9k\alpha^2\sinh k(3\alpha x-2\alpha^2)^{\frac{1}{2}}}{4(3\alpha x-2\alpha^2)^{\frac{3}{2}}}+\frac{9k^2\alpha^2\cosh k(3\alpha x-2\alpha^2)^{\frac{1}{2}}}{4(3\alpha x-2\alpha^2)}$$

$$>0,$$

since $$\tanh k(3\alpha x-2\alpha^2)^{\frac{1}{2}}<k(3\alpha x-2\alpha^2)^{\frac{1}{2}}.$$

We also see that at $x=\alpha$ the left and right-hand first order derivatives of $f(x)$ are $\frac{3}{4}k\sinh k\alpha$ and $\frac{3}{2}k\sinh k\alpha$.

The convexity of $f(x)$ is established.

Queen's University, Belfast R. Cooper

2718. The iterated exponential of x.

Let $x_0 = 1$, $x_{n+1} = x^{x_n}$, $n \geqslant 0$, so that $x_1 = x$, $x_2 = x^x$, and so on, and further let

$$E_{-1}(x) = 1, \quad E_{n+1}(x) = x_{n+1} \,.\, E_n(x), \quad n \geqslant -1,$$

so that $E_n(x) = \prod_{r=0}^{n} x_r$; then

$$Dx_{n+1} = \frac{E_{n+1}(x)}{x} \sum_{p=0}^{n} \frac{(\log x)^{n-p}}{E_{p-1}(x)}, \quad n \geqslant 0. \quad \text{......................(i)}$$

For (i) clearly holds for $n = 0$, and if it holds for $n = k$, then, since $\log x_{k+2} = x_{k+1} \log x$ we have

$$\frac{Dx_{k+2}}{x_{k+2}} = (Dx_{k+1}) \log x + \frac{x_{k+1}}{x}$$

$$= \frac{E_{k+1}(x)}{x} \sum_{p=0}^{k} \frac{(\log x)^{k-p+1}}{E_{p-1}(x)} + \frac{1}{x} \frac{E_{k+1}(x)}{E_k(x)}$$

$$= \frac{E_{k+1}(x)}{x} \sum_{p=0}^{k+1} \frac{(\log x)^{k-p+1}}{E_{p-1}(x)}$$

whence $$Dx_{k+2} = \frac{E_{k+2}(x)}{x} \sum_{p=0}^{k+1} \frac{(\log x)^{k-p+1}}{E_{p-1}(x)}$$

and so by induction, (i) holds for all n.

The derivative of $E_{n+1}(x)$ takes the simple form

$$DE_{n+1}(x) = E_{n+1}(x) \sum_{q=0}^{n} \sum_{p=0}^{q} \frac{E_q(x)}{E_{p-1}(x)} \cdot \frac{(\log x)^{q-p}}{x} \,. \quad \text{...............(ii)}$$

For $n = 0$, the right-hand side of (ii) has the value

$$E_1(x) \,.\, 1/x = 1, \quad \text{and} \quad DE_1(x) = 1.$$

Since

$$\log E_{k+1}(x) = \log E_k(x) + \log x_{k+1},$$

we have

$$\frac{DE_{k+1}(x)}{E_{k+1}(x)} - \frac{DE_k(x)}{E_k(x)} = \frac{E_k(x)}{x} \sum_{p=0}^{k} \frac{(\log x)^{k-p}}{E_{p-1}(x)}$$

whence, by addition, giving k in turn the values 0, 1, 2, ..., n, we find

$$\frac{DE_{n+1}(x)}{E_{n+1}(x)} = \sum_{q=0}^{n} \sum_{p=0}^{q} \frac{E_q(x)}{E_{p-1}(x)} \cdot \frac{(\log x)^{q-p}}{x}$$

as required.

PETER GOODSTEIN

2719. Another pretty series. (Note 2620.)

The series in Mr. Tahta's assertion that

$$\sum_{n=1}^{\infty} \binom{4n}{2n} x^{2n} = \frac{13}{7} \quad \text{when} \quad x = \frac{6}{25}$$

is of a more elementary character than Mr. Ferguson's series. For all values of x whose modulus is less than $\frac{1}{4}$, Mr. Tahta's series is the expansion by the binomial theorem of

$$\frac{1}{2}\left\{\frac{1}{\sqrt{(1+4x)}} + \frac{1}{\sqrt{(1-4x)}}\right\} - 1.$$

It is not without interest to search for positive rational values of x other than $\frac{6}{25}$ which make the sum of the series equal to a rational number. We take

$$\sqrt{(1+4x)} = u/w,$$

where u and w are positive integers with no common factor, and then

$$1-4x = \frac{2w^2-u^2}{w^2}.$$

Accordingly, if $\sqrt{(1-4x)}$ is to be rational, since $2w^2-u^2$ is an integer, it must be the square of a positive integer v which has no factor in common with w; also

$$(u+v)^2+(u-v)^2 = 4w^2.$$

We now write

$$\frac{u+v}{2w+u-v} = \lambda,$$

where λ is a positive rational number, and then we easily find that

$$\lambda^2(2w+u-v)^2 = (2w+u-v)(2w-u+v),$$

$$\lambda^2 = \frac{2w-u+v}{2w+u-v}, \qquad 1+\lambda^2 = \frac{4w}{2w+u-v},$$

$$\frac{u+v}{2w} = \frac{2\lambda}{1+\lambda^2}, \qquad \frac{u-v}{2w} = \frac{1-\lambda^2}{1+\lambda^2},$$

$$\frac{u}{w} = \frac{1+2\lambda-\lambda^2}{1+\lambda^2}, \qquad \frac{v}{w} = \frac{\lambda^2+2\lambda-1}{1+\lambda^2}.$$

Since

$$v = w\sqrt{(1-4x)} < w\sqrt{(1+4x)} = u,$$

we have $u-v$ positive, and therefore $\lambda<1$; while, since v is positive (by definition), we have $\lambda^2+2\lambda-1$ positive and so $\lambda>\sqrt{2}-1$.

Hence, when λ satisfies the inequalities

$$\sqrt{2}-1<\lambda<1,$$

we have both

$$x = \frac{u^2-v^2}{8w^2} = \frac{\lambda(1-\lambda^2)}{(1+\lambda^2)^2}$$

and

$$\frac{1}{2}\left\{\frac{1}{\sqrt{(1+4x)}}+\frac{1}{\sqrt{(1-4x)}}\right\}-1 = \frac{w(u+v)}{2uv}-1$$

$$= \frac{2\lambda(1+\lambda^2)}{6\lambda^2-\lambda^4-1}-1$$

$$= \frac{(1-\lambda)^2(1+4\lambda+\lambda^2)}{6\lambda^2-\lambda^4-1}.$$

Any rational value of λ in the specified range gives these rational values for x and for the sum of Mr. Tahta's series; the value of λ in the numerical example which he considers is $\frac{1}{2}$.

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Editorial Note. Solutions were also received from Mr. B. D. Josephson and Mr. A. K. Rajagopal.

REVIEWS

Mechanik der Kontinua. By G. Hamel, prepared for publication by I. Szabó. Pp. 210. DM 29.70. 1956. (Teubner, Stuttgart)

The substance of this book was in being at the time of Hamel's death, and the responsibility of seeing it through the press was undertaken by Dr. Szabó, a colleague of Hamel's at the famous Technischen Universität of Berlin-Charlottenburg, where Hamel had built up his great reputation as a teacher of applied mathematics.

There are three sections: ideal fluids (pp. 120); viscous fluids (pp. 66); general deformable bodies (pp. 20). The final section is only a sketch, which would possibly have been filled in if Hamel had lived to complete the work. The reader is supposed to be familiar with the algebra of complex numbers and vectors, and with the residue theory and the integral theorems of vector analysis. The main lines of ideal fluid theory are then traced very clearly and compactly, with carefully selected examples and up-to-date references. There are no exercises for the reader, but apart from this defect, as I feel it to be, the work is an admirable first course in hydrodynamics. The section on viscous fluids is so concise that while the mathematics is clear, contact with physical reality is not so well maintained, and this section may perhaps be regarded as setting out only the bare mathematical foundations. Compared with Hamel's big book on *Theoretische Mechanik*, in the Springer " Grundlehren " series, the general lines are here more easily followed, since the small compass leaves less room for the painstaking but almost pedantic discussion of detail which in the larger book at times somewhat obscured a view of dynamic reality. Of the present volume one may say that teachers and students could consult the first section with much profit; the two later sections are a little too condensed for pupils. T. A. A. Broadbent

The theory of games and linear programming. By S. Vajda. Pp. 106. 8s. 6d. 1956. (Methuen)

This little book is in very select company and is only the third or fourth to appear on the rapidly expanding new topics of the theory of games and linear programming. Linear programming is the problem of finding the (non-negative) solution of a set of linear equations which makes the value of a given linear form as small as possible. For instance if $-2x_1+x_2+x_3=2$, $x_1-2x_2+x_4=2$, $x_1+x_2+x_5=5$ then the minimum value of x_2-x_1 for non-negative x_i, $i=1, 2, 3, 4, 5$, occurs when $x_1=4$ and $x_2=1$. An interesting illustration of this solution is obtained by considering x_1, x_2 as cartesian coordinates so that the condition

$$-2x_1+x_2+x_3=2, \quad x_1\geqslant 0, x_2\geqslant 0, x_3\geqslant 0$$

represents the side of the line $-2x_1+x_2=2$ in the first quadrant which contains the origin, and similarly the remaining linear conditions determine regions of the first quadrant; the lines $x_2-x_1=$constant, are all parallel and to solve the stated problem in linear programming we have to find (by inspection) which of this family of parallels passes through a point of the region determined by the conditions (in this case a pentagon) and cuts the axis $x_2=0$ as low down as possible.

The games considered are two person games, like the following card game between A and B. A receives a high or low card with equal probability; on receipt of a high card he must bid £2; for a low card he may pay £1 or bid £2. B may, but need not, challenge a bid; if he does not then he pays £1, but if he challenges he wins £2 if A has a low card and loses £2 if A has a high card. An algebraic formulation of the games problem is shown to lead to a problem in linear programming. R. L. Goodstein

Special Functions of Mathematical Physics and Chemistry. By IAN N. SNEDDON. Pp. 164. 10s. 6d. 1956. (Oliver and Boyd)

This book is a useful addition to a useful series. It contains an account of the functions most important to the mathematical physicist, and carries the theory of these functions as far as most applied mathematicians are likely to require. In these days when most pure mathematicians are chiefly interested in abstract subjects, while the applied mathematician has little time to spare for teaching pure mathematics, this should be very helpful.

The main chapters deal with the hypergeometric, Legendre and Bessel functions and with those named after Hermite and Laguerre. The principal properties of these functions are set out clearly, and examples are given of applications to the problems, old and new, of physics.

As regards notation it seems hardly necessary to introduce the suffixes in the functions ${}_2F_1(\alpha, \beta\,;\ \gamma\,;\ x)$ and ${}_1F_1(\alpha\,;\ \gamma\,;\ x)$. When Barnes introduced this suffix notation the production of mathematical books was much less expensive than now. Today unnecessary suffixes should be avoided in order to simplify the printing. It is probably too late now to change from $P_n{}^m(\mu)$ and $Q_n{}^m(\mu)$ to $P(n; m; \mu)$ and $Q(n; m; \mu)$; but $(\alpha)_r$ is better written $(\alpha; r)$. The notations for $Q_n{}^m(\mu)$ and $T_n{}^m(\mu)$ as given in this book offer no difficulty, because m is confined to integral values; but, in order to avoid complications when m is not integral, the factor $(-1)^m$ in $Q_n{}^m(\mu)$ should be dropped, and a factor $(-1)^m$ should be inserted in the definition of $T_n{}^m(\mu)$.

The printing and format of the book are excellent, and we warmly congratulate both author and publisher. THOMAS M. MACROBERT

Through the mathescope. By C. STANLEY OGILVY. Pp. 162. 24s. 1956. (Oxford University Press)

This is a very charming addition to the Oxford mathematical books for the general reader, and every mathematician who has sought in vain to give his colleagues in the Senior (or Staff) Common Room a glimpse of his subject as a field of discovery and invention will be grateful to Mr. Ogilvy for writing a book which will plead his cause with wit, knowledge and insight. Every schoolboy who hopes to make mathematics his subject at the University should be encouraged to read it.

The author asks (p. 64) for an intuitively obvious proof of the reflection property of the parabola. If light reaches an eye E from a source at the focus F, if Q is any point on the parabola and if the perpendicular from E to the directrix meets the latter at D and the parabola at P, and the perpendicular from Q meets the directrix at R, then

$$FP + PE = DP + PE < RQ + QE = FQ + QE$$

so that FPE is the shortest path from F to E, via the parabola, as was to be proved. R. L. GOODSTEIN

A First Geometry. By A. R. BIELBY. Pp. 168. 6s. 1955. (Longmans)

The results of a recent G.C.E. examination suggest that the introduction of the alternative syllabus has reacted adversely on the teaching of geometry. Many pupils give no sign of having acquired the feel of the subject. Whatever may be the reason for this, a textbook which deals exclusively with geometry is likely to have the advantage over one which offers a general mathematics course, in that its attack can be concentrated, connected and continuous. One's impression of the book under review is that at least it sets a standard which should be helpful in developing the beginnings of geometrical sense. It provides the usual stage A course with sections on loci, similarity, areas and Pythagoras' theorem, and takes deductive work as far as the parallelogram

theorems. Fundamental ideas are stressed in the setting of appropriate practical work and the use of correct terminology and geometrical language are insisted upon from the start.

Much will be required of the teacher by way of discussion, direction and amplification as can be seen, for instance, in the section on field work. The author notes in his preface that the pupil will need help with a first reading and that room has been left in many places for the teacher's fuller explanation. This seems sound policy since the eleven year old is not often disposed to study his textbook. His stimulation and inspiration derive from the teacher whose task it is to secure interest, understanding and appreciation.

The book is concisely written and attractively produced. It merits attention particularly for use with the abler pupil. There is a subject index to help with revision, a summary of content and a list of new vocabulary at the end of each chapter. W. FLEMMING

The Theory of Hydrodynamic Stability. By C. C. LIN. Pp. 155. 22s. 6d. 1955. (Cambridge University Press)

This book is in the series of Monographs on Mechanics and Applied Mathematics published by the Cambridge University Press, and is an admirable survey of the field of hydrodynamic stability written by one of the principal figures in the field.

The precise role of viscosity in the stability of liquid motion has been one of the central problems in hydrodynamics since the middle of the nineteenth century and, apart from the intrinsic interest of the problem stated in this way, the problem involves the onset of turbulence in viscous flows. One facet of this fascinating problem concluded in 1947 with the experimental verification by Schubauer and Skramstad of the theory which had been propounded and developed successively by Rayleigh (1880) Heisenberg (1924) Schlichting (1933) and Tollmein (1929). In this theory, the stability of infinitesimal disturbances in a two-dimensional shearing flow has been investigated and the theory establishes, for a given wave length of disturbance, the range of Reynolds numbers in which this wave length will remain stable and outside of which such a wave will be unstable. Thus viscosity in some ranges may act towards destabilisation. The experimental verification is made difficult due to the fact that the mathematical theory takes the disturbances to the shearing flow to be infinitesimal in amplitude while the Reynolds number for stability is influenced by the finiteness of the amplitude. It may be objected that the theory and the experiment are both rather ideal creations and that the precise conditions required for the verification of the theory are seldom realised in practice but perhaps the principal importance of the verification is that it confirms methods of procedure in the mathematical attack upon this problem which can form the basis for attacks on more complicated problems. The author of the book has, since 1944, succeeded in clarifying the mathematical attacks and this book serves to illumine once more the " crossing substitution ", which formulates the rule for continuation of functions through a particular singular point.

The second facet of this problem, which is also treated in considerable detail, is the stability of the so-called Couette flow of the liquid between two concentric circular cylinders which rotate at different angular velocities. Mathematicians will be well acquainted with the experimental and theoretical investigations of Sir Geoffrey Taylor in this problem (1923) and it is most valuable to have all this material together with all the subsequent contributions available in such a compact form.

Each of the problems mentioned briefly above has stimulated a great number of mathematicians and the book furnishes a complete set of references to

the wealth of achievement in this field. The book is valuable not only for the fact that it has summarised a wide field of research which has existed hitherto in various journals but also for the fact that it achieves what it sets out to perform, namely " to give a reasonable evaluation of the significance and limitations of the results of studies in hydrodynamic stability ".

T. V. DAVIES

Propagation des Ondes dans les Milieux Periodiques. By LEON BRILLOUIN and MAURICE PARODI. Pp. 348. 4000 fr. 1956. (Masson et Cie, Paris)

This book deals with a variety of physical subjects, all of which can make use of similar mathematical methods. A part of it is contained already in Brillouin's book " Wave Propagation in Periodic Structures ", but a substantial fraction of the present book is new and refers largely to Parodi's work.

The first four chapters are devoted to an extensive survey on oscillations of one dimensional lattices. There follows a chapter on general questions of energy flux ; the one dimensional case is completed by an investigation of the properties of finite periodic structures. Chapters 7 and 8 then deal with some two and three dimensional cases. The following chapter is concerned with problems that can be expressed in terms of the Mathieu equation. Four chapters (10–13) then discuss questions of electrical networks, and of wave guides. Finally the motion of an electron in a moving electric field is treated.

Clearly few people have an interest in all the subjects treated in the book but, nevertheless, it should fulfil a useful function. Thus electrical engineers have developed methods to deal with complex networks which might be of use to a physicist dealing with oscillations of complex systems. It is of interest, in this connection, that on page 233 the authors present the reader with a dictionary to help translation from mechanical into electrical " language " (e.g. potential difference = tension).

Amongst the subjects that have not been considered by the authors is a treatment of anharmonic oscillation, a subject which is of considerable interest to physicists (e.g. conduction of heat in solids), and which often meets with formidable mathematical difficulties.

The book can be recommended to those who wish to study in detail and to apply the mathematical methods used in investigation of the wave propagation in periodic structures of all kinds.

H. FRÖHLICH

Groups of Finite Order. By R. D. CARMICHAEL. Pp. 447. $2, paper bound, $3.95 cloth. 1956. (Dover Publications, New York)

The chapters on finite geometries make this a very useful addition to the Dover reprint series. Carmichael's text is deservedly famous for its lucidity and progress by easy stages from permutation groups to Galois fields.

R. L. GOODSTEIN

Geometry of Four Dimensions. By H. P. MANNING. Pp. ix + 348. $1.95. 1956. (Dover, New York)

The analytical theory of heat. By J. FOURIER. Pp. xxiii + 466. $1.95. 1956. (Dover, New York)

Manning's Geometry is an unabridged and unaltered reprint of the 1914 edition, well produced and quite good value for about 15s. ; a cloth bound edition is available for an extra $2.

Fourier's theory of heat is of course only of historical interest now, but this reprint of the 1878 translation of Alexander Freeman makes available again a book every mathematician has read and delighted in. What student has not heard of the lengthy calculations by which Fourier obtained the coefficients in the first Fourier expansions, and then observed (p. 187) that the result could have been obtained simply by term by term integration.

R. L. GOODSTEIN.

Intuitionism. An Introduction. By A. HEYTING. Pp. viii + 132. 24s. 1956 (North Holland Publishing Company, Amsterdam)

This is a book which all who are interested in foundation studies have awaited for a long time. Such an authoritative account of intuitionism written with charm, clarity and well balanced judgement will go a long way towards dispelling the irrational fears with which intuitionism is met by many working mathematicians. We see intuitionism, not as a dragon guarding secret rites and mysteries, not as a weapon to cut the heart of mathematics, but as a new system of mathematics full of interesting unexpected results.

The greater part of the book is devoted to the development ofintuitionistic analysis, including a theory of measure and Lebesgue integration carried as far as a proof of the Riesz-Fischer theorem. There is also a chapter on algebra and a chapter on logic.

In intuitionistic analysis any function defined in a closed interval is uniformly continuous in the interval and any bounded function which is defined almost everywhere is measurable, which is of course in striking contrast with the familiar situation in classical analysis. It is an obvious consequence that a function $f(x)$ cannot be defined by such conditions as

$$\begin{aligned} f(x) &= 0 \quad \text{if } x \neq 0 \\ f(0) &= 1 \end{aligned}$$

and that this is *not* a definition of a function in intuitionistic analysis is perhaps more remarkable than any other situation in the system. It is not in fact difficult to see why intuitionism must exclude such a definition. There are convergent sequences (of rationals) a_n of which it is impossible to say whether a_n converges to zero or not and yet such convergent sequences are real numbers. If ξ is such a real number, the value of $f(\xi)$ must be determined by the sequence of values $f(a_1)$, $f(a_2)$, $f(a_3)$, ..., and in no other way, and this obliges $f(x)$ to be continuous at ξ.

In algebra, intuitionistic requirements lead to greater precision. For instance if it is not known whether a is zero, or not zero, and not known whether b is zero or not zero, and if further the proportion of a to b is not known, (as for instance when $a=(a_n)$ and $a_n=2^{-n}$ if for all $r\leqslant n$, $2r$ is a sum of two primes and $a_n=2^{-k}$ if k is the first number less than $n+1$ such that $2k$ is not a sum of two primes, and $b=(b_n)$ where $b_n=3^{-n}$ if $a_n=2^{-n}$ but $b_n=3^{-k}$ if $a_n=2^{-k}$) then it is not known whether the equation $ax+by=0$ has a solution, other than $x=y=0$, or not.

The key theorem in the book is the fan theorem which says, in effect (to use classical language) that if $f(x)$ is any real function defined in an interval (a, b) there is an integer function $N(n)$ such that for any ξ in $(a, \ b)$ the value of $f(\xi)$ to $N(n)$ places of decimals is determined when ξ is known to n places of decimals. The reviewer finds the proof of this theorem far from convincing and would suggest taking the result as a postulate since it seems to be too closely part of the very definition of function in intuitionism to be susceptible of proof in a non-formalised system of mathematics.

R. L. GOODSTEIN

Cours de Cinématique. Volume II, 3rd Edition. By RENÉ GARNIER. Pp. x, 341. 5000 fr. 1956. (Gauthier-Villars, Paris)

Volume I of this work (reviewed in the *Mathematical Gazette*, 1956, p. 64) contained an account of the elementary notions of kinematics, the treatment being based on a combination of vector analysis and the method of moving frames. In this second volume these tools are applied systematically and in great detail to the problem of rolling motion and the more general problem of "viration". The new edition differs from previous ones by the addition of new material (including four notes at the end of the book extending to 25 pages), and also by the greater emphasis which is placed on geometrical methods and arguments.

One of the main topics considered is the extension of Savary's formula to three dimensions and its applications. Most readers will find that the very detailed treatment given in this second volume makes it more arduous reading than the first. It is also doubtful whether the book will be widely read in this country, not because of language difficulties but because there is no tradition for the extensive study of the subject. In France, however, the contributions of Cauchy, Savary, Chasles, Poncelet, Poinsot, Serret and others have ensured the subject a more prominent position in mathematical education.

T. J. Willmore

Encyclopaedia of Physics. Edited by S. Flugge. Mathematical Methods I Pp. 364. DM 72. 1956. (Springer, Berlin)

The opening article in this volume, on the concepts of classical analysis, ordinary differential equations and complex function theory is by J. Lense. It is beautifully written, full without being overwhelmed by detail, thorough and exact. As a trifling indication of its carefulness one may mention the integrals

$$\int \frac{1}{x}\, dx = \log |x|, \quad \int \frac{1}{\sin x}\, dx = \log |\tan \tfrac{1}{2}x|$$

with their positive value signs so frequently overlooked. A slight blemish was found in the statement (p. 31) of the conditions for substitution in the definite integral where the condition that $f(\phi(t))$ must be continuous in (α, β), in the case when $\phi(t)$ takes values outside (a, b) for a t inside (α, β), was omitted. The article concludes with a note on the theory of measure and Lebesgue integration.

The second article, by the same author, is on partial differential equations, and deals with a wide range of topics; Jacobi multipliers, Poisson, Jacobi and Lagrange brackets, Pfaffians, infinitesimal transformations, absolute and relative integral invariants. The third article, also by Lense, deals with elliptic functions and integrals and after a brief enumeration of the general properties of elliptic functions the development follows the familiar order; the Weierstrass function defined by its partial fraction expansion, and its inverse function as an integral, the zeta and sigma functions, addition theorems, Riemann surface and conformed mapping, the elliptic modular function, the Jacobian functions and the reduction of elliptic integrals to standard forms.

The section on the special functions of mathematical physics is by J. Meixner; this section is remarkably complete and comprehensive and yet it is more than a recital of facts. The article is organised under three principles; after definition by infinite series special functions are studied as solutions of differential equations, by functional equations and by difference equations. The whole treatment is unified by means of Truesdell's solution of the equation

$$f(x, \alpha+1) = dF(z, \alpha)/dz, \quad f(z_0, \alpha) = \phi(\alpha)$$

in the infinite series

$$f(z, \alpha) = \sum_{n=0}^{\infty} \phi(\alpha+n) \frac{(z-z_0)^n}{n!}$$

(when $\lim \sup\{|\phi(\alpha+n)|/n!\}^{1/n}$ is finite).

The final section, on boundary value problems by F. Schlogl, is both the longest and most extensive and varied in the book. It opens with a brief account of Fourier series and Transforms, using the Lebesgue integral and the theory of convergence in mean. Next comes an introduction to linear transforms in functional space with a reference to the Dirac δ-function. Linear integral equations are studied in some detail with an account of the solution in a

Neumann series. Another topic treated in detail is the calculus of variations with an application to the Euler-Lagrange equation. The section concludes with an attractive account of boundary value problems in second order partial differential equations; the characteristics of hyperbolic, parabolic and elliptic equations, the determination of Green's function in the Sturm-Liouville eigenvalue problem and finally the Laplace equation $\Delta\psi - a^2\psi = 0$.

R. L. GOODSTEIN

Encyclopedia of Physics. Volume II. Mathematical Methods II. Edited by S. FLUGGE. Pp. 520. DM 88. 1955. (Springer, Berlin)

This volume of the Encyclopedia contains five articles, four of which are written in German, and the fifth, by Sneddon, in English. Sneddon's contribution is the longest and covers Laplace, Fourier, Mellin and Hankel transforms, with a brief introduction to Lebesgue integration theory and Banach space. There are also *brief* sections on Schwartz's theory of distributions and variational methods in functional analysis.

The section on Algebra, by Falk, is about a hundred pages long, and is concerned entirely with modern concepts; this article is no mere enumeration of results but presents an interesting and detailed development of group theory, polynomial rings, matrix theory, with application to quadratic and Hermitean forms. The geometry section by Tretz, (nearly a hundred pages in length) covers projective and differential geometry, vector field theory, tensors and spinors.

The section on numerical and graphical methods is by Colletz; this section contains chapters on nomograms, iterative solutions of equations (on an original theoretical basis), Horner's method and Graeffe's root squaring process, finite differences, interpolation and numerical integration (with a table of the error terms in some dozen different methods), and the numerical integration of ordinary and partial differential equations. There are also a few paragraphs on the handling of functional equations. The last section, by Buckner, is on computing machines, and deals briefly with the theory of programming for a digital computing machine.

R. L. GOODSTEIN

Selecta Hermann Weyl. Pp. 592. 1956. (Birkhauser, Basel)

When the greatest mathematician of his generation died on the 8th December, 1955, this magnificent volume in honour of his 70th birthday, produced jointly by the Zurich Federal Institute of Technology and the Princeton Institute of Advanced Study, was passing through the press.

The selection of 20 papers from the 160 which Weyl published was made by Eckman, Hopf and Plancherel in Zurich and Morse, von Neumann and Selberg, in Princeton. The papers have naturally been chosen to show Weyl's range and versatility and cover a period of some forty-two years. There are papers on potential theory, on Electromagnetic waves, the properties of elastic bodies, the foundations of mathematics, the theory of continuous groups, quantum mechanics, division algebras, interpolation, analytic number theory, integral equations, spinors in n dimensions and the theory of harmonic integrals.

The paper and printing are of a fine quality all too rarely seen today and worthy of the material they display.

R. L. GOODSTEIN

The Theory of Functions of Real Variables. By L. M. GRAVES. 2nd Edition. Pp. 375. 56s. 6d. 1956. (McGraw-Hill)

The first edition of this book was reviewed some ten years ago in XXXI, 184 by J. H. Pearce, who praised the author for his accuracy and deplored his unhelpful style. The main change in this revised edition is the addition of two new chapters; the first of these treats the general theory of sets, relations and functions, ordinal and cardinal numbers, the axiom of choice and some equivalent propositions. The treatment of set theory is Cantorian and there is no

mention of the paradoxes. The coded telegram style of the earlier chapters is maintained; we are told in the preface that a properly prepared student may go immediately to this chapter after a brief look at Chapter I. Apart from reading the same material in a more leisurely text one wonders what would constitute a proper preparation. The other new chapter is on metric spaces and deals with complete spaces, separable spaces and compact spaces and sets.

R. L. GOODSTEIN

Proceedings of the second international congress of the internation union for the philosophy of Science. Vols. I–V. Pp. 774. 38 Fr. S. 1955. (Griffon, Neuchatel)

The first volume of these Proceedings contains the lectures given in plenary sessions; the third and (largest) volume contains the lectures on linguistics and the theory of knowledge, the fourth is concerned with the history and philosophy of science and the fifth with psychology and sociology. Only the second half of the second volume contains lectures on specifically mathematical topics; amongst the contributors to this volume are LORENZEN on an extension of the field of results attainable by finitist methods, ROSENBLOOM on constructive equivalents of theorems of classical analysis and TEENSMA on the intuitionistic interpretation of analysis. The five volumes contain in all 101 lectures. R. L. GOODSTEIN

Logic, Semantics, Metamathematics. By A. TARSKI. Translated by J. H. Woodger. Pp. 471. 60s. 1956. (Clarendon Press)

This volume contains translations of Tarski's papers on mathematical logic published between the wars, with additional notes by the author referring to later developments and recent literature. The important part which the Polish school of logicians played in foundation researches is well known and the publication of this work is as much a tribute to them, and to those like Dr. Lindenbaum who fell a victim to the Gestapo during the war, as to the Author himself.

The articles collected together vary from quite short reports in general terms to the long papers on the concept of truth in formalised language and the fundamental concepts of the methodology of deductive sciences. Tarski's paper on the concept of truth is more often referred to than read and the present publication in English should do much to remove the widespread suspicion that Tarski's theory is essentially trivial. Certainly a definition of the form

the sentence x is true if and only if p

where "p" is replaceable by any sentence of the formalised language in question and "x" is simultaneously replaceable by the name of this sentence, at which we arrive after 35 pages has little of novelty to recommend it, but there is much more in the article than this. In addition to the semantic definition there is a search for a structural definition which proves successful in the case of the language of the calculus of classes where it is shown that suitable additions to the axioms (which ensure that the class of all individuals in infinite) bring the class of true sentences into coincidence with the class of provable sentence. Such a structural definition is, in view of Gödel's theorem, impossible in recursively axiomatizable language rich enough to contain recursive arithmetic, for in such a language there will always be sentences true under the intended interpretation of the system but which are not provable under any recursive extension of the axioms.

Another of the major papers in the volume is that on the fundamental concepts of the methodology of deductive sciences which contains a formalisation of the metatheory of a formal system independent of (and perhaps prior to) Gödel's, but as Tarski himself says (in a later paper) the results he achieved are almost trivial beside Gödel's proof of the incompleteness of arithmetic.

Tarski's work makes no contribution to finitist trends in foundation researches but it forms the basis of recent work by Robinson and Henkin on the metamathematics of Algebra.

The long delay in the publication of this book is ascribed to the tragic death (in 1953) of Jan Kalicki in a motor accident in a car which Tarski was driving. Kalicki came to England as a British Council Scholar some ten years ago and worked as a research student with the late Paul Dienes. He published important papers on many-valued logic and was appointed to a lectureship in Leeds; later he joined Tarski in California. This charming and brilliant young man lost his life just as his star was shining most brightly and it is fitting that he should be remembered by the tribute which Tarski pays to him in the preface.

R. L. GOODSTEIN

Progress in Nuclear Energy. (Editors R. A. CHARPIE, J. HOROWITZ, D. J. HUGHES and D. J. LITTLER). Series I, Physics and Mathematics, Vol. 1. Pp. 400. 70s. 1956. (Pergamon Press)

"Scientific literature by tradition is based on results without regard to the nationality of the authors. In fission physics, however, the events of the last decades had so far prevented the realisation of this normal state of affairs. Let this volume be a symbol of a return to normalcy in scientific relations." Professor V. Weisskopf thus emphasises, in his Foreword, the exceptional circumstances of the publication of the series "Progress in Nuclear Energy" of which this volume is the first to be released. It represents a critical compilation of work which had been released by the various countries at the Geneva Atomic Energy Conference in August 1956, together with additional information collected since the conference from experts in particular fields of work, namely branches of the physics and mathematics of the fission process. There is one Russian author, and considerable use has been made of Russian material in some of the other articles.

This first volume of the Physics and Mathematics series starts with the most basic of all fission data, the neutron cross-sections of the fissionable elements for the various processes possible. A very large amount of work on this subject has been done, and from it the details of the fission and capture cross-sections have been established with a high degree of accuracy up to neutron energies of the order of 1 or 2 ev. The results are set out in an article by Harvey and Sanders. This is the important region for thermal reactors, but one hopes that one day equally comprehensive data for higher energies will become available. At these energies at present results are only fragmentary, and H. A. Bethe has made an attempt at semi-theoretical estimates in his review article. One experimental fact is of great importance; it is that the ratio of radiative to fission capture is high for Pu_{239} and U_{235} in the whole region below 10^5 ev. Hence there is very little hope of an intermediate energy breeder using these fissile elements. We must stick to high neutron energies.

In another comprehensive article, Codd, Shepherd and Tait describe both the theory of fast reactors and the results obtained with experimental low power reactors. The theoretical part of the article gives a full account of the calculation methods which can be used for evaluating the neutron density in such a reactor, and the perturbations due to small changes in structure. This is a quite complex mathematical exercise. It perhaps looks a little academic, since not much attempt is made to use it in interpretation of the experimental results, on account of the fact that the various cross-sections to be inserted are not well enough known. However the article does give a great deal of information on the behaviour of those complex systems known as fast reactors. For instance the feasibility of breeding by such means is definitely established.

Perhaps calculation methods may be better applied to thermal reactors. In

the only Russian article, S. M. Feinberg in fact does this for the heterogeneous type of reactor, with results in quite good accord with experiments, even in the case of experimental assemblies containing only a small number of uranium rods. But whether the method could be applied accurately to an actual reactor, with its mass of holes and control rods in the core, may be uncertain.

Other articles deal with various other physical problems of reactors and are mainly experimental.

I think this volume gives an excellent start to the series, and it is to be hoped that other volumes will follow it before long, so that we shall have as comprehensive a set of review articles on the various problems of Nuclear Energy as we have in other branches of Physics and Mathematics. H. W. B. SKINNER

Neue Topologische Methoden in der Algebraischen Geometrie. By F. HIRZEBRUCH. Pp. viii, 165, 1956. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 9. (Springer, Berlin)

The title of this work prompts the reflection that twice during the present century the application of topological methods to algebraic geometry has revolutionised the subject. The first revolution was effected over thirty years ago by the systematic study made by Lefschetz of the homology groups of algebraic varieties, and may be said to have been firmly established with the publication of his classical Borel tract " L'analysis situs et la géométrie algébrique " in 1924. The development of the subject thereafter pursued an evolutionary course for a quarter of a century, the most notable contribution (on the topological and transcendental side) being the development by Hodge of the theory of harmonic integrals. During the last few years a second revolution has taken place, of which the work under review gives the first connected account. Two main streams seem to contribute to the latest development; the Cartan-Leray theory of stacks (faisceaux), which has been exploited with great success, notably by Spencer and Kodaira, and the modern theory of fibre bundles, which plays a major part in the recent investigations of the author, which are here described in detail for the first time. The striking success which has attended the first applications of modern topological techniques to algebraic geometry has opened up new horizons in the subject, and it is clear that the way is open to further important advances. In the new era which has dawned the work under review will serve the same purpose as Lefschetz's tract did thirty years ago. Its publication is a major mathematical event.

It is impossible within the space of this review to do full justice to the rich material comprised in these 160 pages. At least half the book is devoted to essential background material, all of which deals with notions unheard of fifteen years ago: only a handful of the fifty odd references are to works published before 1950. An opening paragraph deals with certain formal concepts (multiplicative sequences), whose very skilful exploitation by the author forms one aspect of his own contribution to the theory. There follows, in some 20 pages, a conspicuously well-written account of the theory of stacks, which will be an invaluable introduction to the subject for the classically-minded geometer. Next there is a short account of the relevant parts of the theory of fibre bundles. This is well documented with references to Steenrod's book, but the reviewer, at any rate, would have been grateful for one or two further references. Next comes an introduction to the Chern and Pontriagin classes (based on an axiomatic approach). The second chapter gives a clear account (without detailed proofs) of Thom's theory of " cobordisme ", leading up to Thom's fundamental result that two differentiable manifolds have the same Pontriagin numbers if, and only if, some multiple of their difference is a bounding manifold. This leads to the first of the author's own results (and a key result in further developments),

namely, that the " index " of an oriented compact complex manifold can be expressed in terms of the Pontriagin numbers.

The author's own principal contribution to the theory is a very general theorem expressing the characteristic $\Sigma(-1)^i \dim H^i(V, W)$ of a vector space bundle W over an algebraic variety V in terms of the Chern classes of W and V; here $H^i(V, W)$ is the i-th cohomology group of V with coefficients in the stack of locally holomorphic sections of W. This result appears as the climax of a long and intricate argument, which combines some ingenious formal handling of multiplicative sequences with applications of a number of deep topological theorems. His result includes, as special cases, a general form of the Riemann-Roch theorem for varieties of any dimension, and a proof that the arithmetic genus of an algebraic variety can be expressed in terms of the Chern numbers. The third chapter of this work deals mainly with formal preliminaries required for the proof of this theorem, but includes a non-formal section dealing with properties of so-called " split " manifolds. The early part of the fourth (and last) chapter gives an account of Kodaira's work, and might perhaps have been developed a little less concisely. The book concludes with the proof of the author's main theorem. There is a useful bibliography and a good index.

This well-written work (which seems, incidentally, to be practically free from misprints) will obviously be compulsory reading for the modern algebraic geometer. It is hard going, because it deals with a large complex of important and difficult ideas; but the author's presentation is notable for its clarity, and the book will repay close and detailed study. We may be grateful to the author and his publishers for the production of this notable contribution to mathematical literature.

J. A. Todd

Investigations on the theory of the Brownian movement. By A. Einstein. Pp. 122. \$1.25. 1956

The principles of mechanics. By H. Hertz. Pp. xlii, 274. \$1.75. 1956

Hydrodynamics. By H. L. Dryden, F. P. Murnaghan and H. Bateman. Pp. 634. \$2.50. 1956. (Dover Publications, New York)

Three more classics of applied mathematics are now available in the famous series of Dover reprints.

Einstein's five papers on the Brownian movement, written between 1905 and 1908, were translated and collected into book form in 1926, but perhaps a little overshadowed at that time by the triumphs of his relativity theory. But recent study of stochastic processes may well revive interest in these short but masterly contributions to the subject.

If some 19th century scientists saw the universe in terms of dynamical models, there were critics who insisted that such views could not be securely held until the foundations of dynamics had been safely laid. Do we need force or energy as a fundamental concept, or can we take space, time and mass as the only primitive ideas, in effect reducing dynamics to a kind of kinematics? Hertz worked on this last line, and his influence can be readily traced into the relativity era. The text itself is austerely set out in rules, propositions, corollaries and the aridity of axiomatics, but by frequent reference back to Hertz's own introduction, von Helmholtz's preface, and a new introductory essay by R. S. Cohen, the patient reader will reap a valuable harvest.

The third item is a reprint of the well-known but not too easily accessible Bulletin 84 of the National Research Council, U.S.A., published in 1932. Dryden surveys aerodynamic theory and experiment, Murnaghan deals concisely with the hydrodynamics of a perfect fluid, but the bulk of the book (pp. 89–601) is by Harry Bateman, on viscous fluids and turbulent flow. It exhibits Bateman's insight into principles combined with his mastery of detail and command of the literature.

T. A. A. Broadbent

Ordinary non-linear differential equations in engineering and physical sciences. By N. W. McLachlan. 2nd edition. Pp. x, 271. 35s. 1956. (Clarendon Press, Oxford; Cumberlege, London)

Dr. McLachlan has always insisted that in these days engineering requires competent mathematicians, and his numerous books have endeavoured very successfully to make up-to-date mathematical methods available to the engineer by presenting them through the medium of problems to be solved. His book on non-linear differential equations, first published in 1950, was a pioneering work, and the new edition incorporates material which has been found of value in courses given by the author in American universities. One new chapter gives a geometrical " phase plane " account of singular points and stability criteria, another deals with non-linear problems in fluid flow in two dimensions. Other new material has been worked into the existing chapters; and one very valuable new item is the collection of 52 problems, with solutions, for in a field not yet exploited by the examiner problems of just the right shade of difficulty to attract and not frighten the novice are not easy to come by. The bibliography has been brought up to date; I wonder if in a third edition, in which no doubt the bibliography will be still further extended, Dr. McLachlan would consider attaching distinguishing marks as a help to further study. T. A. A. Broadbent

Numerical Analysis. With Emphasis on the Application of Numerical Techniques to Problems of Infinitesimal Calculus in Single Variable. By Z. Kopal. Pp. xiv, 556. 63s. 1955. (Chapman & Hall, London)

In this book one of the leading numerical analysts of the present day gives a systematic and detailed treatment, from a point of view intermediate between that of the practical computer and that of the pure mathematician, of the branches of numerical analysis which may be broadly described as polynomial interpolation, numerical differentiation, integration of ordinary differential equations (including methods of solving boundary-value problems), mechanical quadratures, and numerical solution of integral and integro-differential equations. The appendices include some useful tables. The result is a treatise of greater originality than one would normally expect in the case of such a subject; it contains both formulae not widely known, and material due to the author (despite his modest omission of his name from the indexes). The general level is perhaps most strikingly illustrated by the author's confession that among the problems at the ends of the various chapters are a few " to which he would like to know the answer himself ".

The book is well printed and produced. A few slips and misprints (even in formulae) and occasional mis-spellings seem hardly worth mentioning when considered in relation to the magnitude of the work. The tripartite system of references (e.g. equation " VII - D - 8 "), though logical, is a little cumbersome in use. There is no general agreement about notation in finite differences, so that failure to please everybody is inevitable. The English is highly creditable to the author, even though in one or two places it would not be recognized as correct on either side of the Atlantic. Deviation from standard English is moderately frequent, but the maximum deviation is small; what better tribute could be paid to an expert on approximation?

Whatever its detailed merits and faults, this is surely a very noteworthy publication. No mathematician with a special interest in the subject is likely to regret buying the book. However much he knows about numerical analysis in general, he can be fairly certain of finding some topic on which Professor Kopal knows more.

How far the book may be regarded as suitable for the general student of mathematics depends on what the general student is expected to know. The growth of numerical analysis indeed raises a teaching problem, which the book under review exemplifies. The field of the book is sharply delimited; it makes

no claim to be a general treatise on the whole subject of numerical analysis. It excludes topics such as the numerical solution of algebraic equations of higher degree, and of systems of simultaneous linear equations, on the ground that these topics have been written up frequently and well elsewhere. It excludes treatment of functions of two or more variables, and therewith the numerical solution of partial differential equations. But within its limits, the book expounds and develops such a wealth of formulae and methods, with adequate numerical examples and a considerable bibliography, that it runs to a number of pages sufficient to daunt some students. The author says that " the book contains more material than can be conveniently presented in a full year's course " and that " in general, its level is such that it can be used with profit by undergraduate students of numerical analysis from approximately the junior level of American colleges and universities ; elementary calculus and some algebra being the only really necessary pre-requisites ". One wonders how far a full course in numerical analysis, including the topics left aside by Professor Kopal, and presumably extending over something like three years, is likely to be available, and ought to be available, to the generality of undergraduates reading for an honours mathematical degree in this country, even in universities which have already given numerical analysis an important place in their honours courses. It is possible that a full treatment of the subject may have to be reserved for undergraduates specializing in numerical analysis or for graduates engaging in research or taking a postgraduate diploma course in numerical analysis. One is reminded of the study of dynamics some sixty years ago, when the dynamical enthusiast might perhaps work through Routh's *Rigid Dynamics*, but the average serious student needed some more concise treatment of the subject. If the teacher of numerical analysis does not treasure his Kopal for as many decades as the teacher of dynamics treasured his Routh, it will only be because of the accelerated rate of scientific progress. A. Fletcher

Automatic Digital Calculators. By A. D. and K. H. V. Booth. 2nd ed. 32s. 1956. (Butterworths, London & Academic Press, New York)

It is a surprising fact that, although digital computers have attracted very widespread interest and much effort has been devoted to their design, the number of books dealing with the subject is still quite small. The present book is almost unique in having reached a second edition, the first edition having been published in 1952.

It is usual to discuss digital computers under the headings, " logical design ", " circuit design ", and " programming ". The present book is addressed to those who are interested in all of these subjects, but people who wish to know how digital computers are used, without paying much attention to how they are constructed, will find their needs met by reading the three short introductory chapters followed by chapters 13–17.

For the second edition new sections dealing with transistors, matrix storage, and automatic coding procedures have been written and a few extra paragraphs have been added to certain chapters. Otherwise there has been less revision of the text than might have been looked for in view of the rapid development in the subject. In several cases revision has taken the form of replacing the date 1952, where it occurred in parenthesis after some such phrase as " at the time of writing ", by 1956, without change in the accompanying text. In one case (page 18) the statements made in the text were true in 1952 but are no longer so.

It is not to be expected that in a book of 250 pages an exhaustive treatment of a large and in some places controversial subject could be given. The book does, however, provide a good general introduction to the subject which leaves few aspects unmentioned. There is an extensive classified bibliography which will enable readers who wish to do so to carry their studies further.

M. V. Wilkes

Rectangular-Polar Conversion Tables. Designed and compiled by E. H. NEVILLE. Pp. xxxii, 109. 30s. 1956. Royal Society Mathematical Tables, Volume 2. (Published for the Royal Society at the University Press, Cambridge)

The arguments of the main table, which occupies pages 1 to 105, are the pairs of positive integers x, y such that $y \leqslant x$, $x \leqslant 105$, $y \leqslant 105$. If r and θ denote $\sqrt{(x^2+y^2)}$ and $\tan^{-1}(y/x)$ respectively, the table gives $\ln r$ and θ in radians, to 15 decimals, and r, and θ in degrees, to 13 decimals. The construction of the tables (by many hands) is described at length. The table was planned in order to provide table-makers with accurate values at the lattice points mentioned, and the results may be regarded as definitive.

Although interpolation was not primarily intended, and the compiler points out that rectangular-polar conversions can in many cases be effected directly with the aid of other standard tables, he proceeds to demonstrate that interpolation is, in fact, much easier than might have been expected. Some small supplementary tables given on pages 106 to 109 are useful, but no tables outside the volume are required. For any given x and y $(0<y<x)$, the key is to locate y/x in the correct interval between two successive terms a/b and c/d belonging to the Farey series of order 105 (that is, the series of proper fractions in their lowest terms with denominators not exceeding 105, arranged in order according to the values of the fractions); the denominators (only) are set out on page 107. Since the Farey series in question has 3375 terms, whose values are distributed between 0 and 1 with an approach to uniformity, the successive terms a/b and c/d bracket y/x fairly closely. Thus corrections to the tabulated values may be calculated without difficulty; for example, angular corrections may be evaluated by using Gregory's series for $\tan^{-1}t$, where t is quite small.

The introduction is graced by the compiler's customary distinction of style, and the whole volume by the usual Cambridge excellence of production. A reference to Volume 1 of the same series is relevant. That volume, which appeared in 1950, was likewise designed and compiled by Professor Neville, and was entitled *The Farey Series of Order* 1025. It was reviewed in *Math. Gazette*, 36, 60, 1952.

A. FLETCHER

Infinite sequences and series. By KONRAD KNOPP. Translated by F. Bagemihl. Pp. 186. $1.75. 1956. (Dover Publications, New York)

This volume must be regarded as a scoop by Dover Publications. In 1922 Knopp wrote the finest text book on infinite series since Bromwich's subtle genius set the style. The present book is no mere shortened version of the larger book but has been completely rethought and replanned to absorb modern material. As far as it goes, it is an ideal text for the younger undergraduate. It is easy to read, absolutely reliable and packed with information. One of the most striking novelties is in the treatment of the Gauss test for convergence of complex series; apart from Pringsheim's Encyklopädie volume no other book, as far as I know, not even Knopp's own treatise, contains a proof of the complete test. Bromwich for instance gives a reference to an early volume of the *Archiv. für Mathematik*. Thanks to a recent discovery by H. Jehle, Knopp is able to cover in full detail the complete Gauss test for positive, alternating, complex series and power series in some ten pages. Another striking feature is the range of powerful theorems on double series which successfully dominate the theory of the multiplication of series.

The English student (and his teacher) will regret the absence of serious examples, but perhaps a little volume of examples is being prepared. It is a very great pity that the book stops short of the theory of uniform convergence. For want of it some proofs take on an unfamiliar guise, for instance Abel's limit theorem. I think many teachers would willingly dispense with the last two chapters on the expansion of the elementary functions and numerical evaluation in return for one or two chapters on uniform convergence.

R. L. GOODSTEIN

The Structure of Turbulent Shear Flow. By A. A. Townsend. Pp. 316 + xii. 40s. 1956. (Cambridge University Press)

The turbulent or irregular motion of fluids remains one of the branches of classical mechanics which presents many unsolved problems to the mathematician. Since 1868, when Reynolds in Manchester described his observations on the disintegration of smooth laminar flow into an irregular eddying motion as the speed of flow increases, the phenomenon has invited, and received the constant attention of experimenters and theoreticians. The mathematical difficulties are formidable, the basic partial differential equations of motion being non-linear and the statistical equations indeterminate, and so inspiration has been sought in careful observation.

The three stages of evolution of a physical theory are first the observation of the phenomenon, then the interpretation and finally the subsequent development and formalization of a mathematical theory. These phases are clearly not distinct; new observations may necessitate revision of the mathematical structure, which may in turn suggest further measurements. In turbulence, all three phases have received attention, but the interpretation and mathematical development have not been free of troubles—the history of the subject is marked by the remains of theories, only partially successful and discarded. Many of these were based upon analogies with the kinetic theory of gases, likening the random motion of the eddies to that of the gaseous molecules. The analogies were obviously imperfect and detailed measurements soon showed their shortcomings. It became evident by about 1945 that the problem was much more complex than the old theories envisaged, and that the need was for a much firmer observational basis before a new theory could be developed.

In the last ten years, this background has largely been filled in, both by Dr. Townsend in England and by a number of workers in the United States, and this book is an attempt to interpret this information and to develop from it a consistent and logical theory. The author is concerned largely with establishing the large scale structure (in a statistical sense) of the various turbulent motions, such as those in wakes and jets, and in pipes and channels. These ideas, combined with the equations of motion, allow a description of the processes which take place in the turbulence and an analytical and determinate formulation of the problem.

The book is for the most part clearly written and enlivened with a quiet, dry humour. Much of the work that it contains has not previously been published and is of the utmost importance to specialists in the subject. But it should also appeal to the non-specialist, to those interested in the evolution from systematic observation to fully developed mathematical theory. O. M. Phillips

Lectures on Partial Differential Equations. By I. G. Petrovsky. Translated from the Russian by A. Shenitzer. 41s. 1954. (London: Interscience Publishers Limited)

Partial Differential Equations. By G. F. D. Duff. Pp. x + 248. 35s. 1956. (Toronto University Press; London: Geoffrey Cumberlege)

The theory of partial differential equations has been somewhat neglected of recent years in this country. The two books under review are important additions to the scanty literature of the subject in English; one is by a distinguished Russian mathematician who has made important contributions in this field; the other by a younger man at Toronto.

Professor Duff's book is intended to provide the student "who has a good background in ordinary differential equations" with a coherent account of the theory of first and second order partial differential equations and systems of such equations. Much of the work is classical, and will recall to the reader Goursat's Cours d'Analyse and the second volume of Courant-Hilbert; but

it is not all classical—it introduces the reader to many modern developments.

The book by Professor Petrovsky which had its origin in his courses of lectures at Moscow State University, was first published in 1950 in Russian. The English edition is the translation of the first Russian edition. A German edition published by Teubner in 1955, appears to be the translation of the second Russian edition of 1953. There is, however, little difference between the two translations.

Petrovsky's book, being a course of lectures, is not so systematic or well-balanced as Duff's, and is on a more elementary level. He makes no attempt to prove theorems in their greatest generality. He is content to give rigorous proofs for the simplest types of equation and to indicate how the ideas can be extended. Professor Duff, on the other hand, considers the general cases with an arbitrary number of independent variables, and makes use of the ideas of the tensor calculus and Riemannian geometry. The subject is difficult enough in any case; and I personally prefer to encounter difficulties one by one and to start with easy cases even though generalisation may later involve a certain amount of repetition.

Professor Petrovsky's book is divided into four chapters of very differing lengths; that the longest is on equations of hyperbolic type reflects his special interests.

In the first introductory chapter of 61 pages, the Cauchy-Kowalowsky existence theorem is proved. This leads to the idea of a characteristic surface and to the classification of equations of the second order with one dependent variable. The Chapter concludes with the reduction of a quasi-linear equation of the second order with two independent variables to canonical form.

Chapter II, of 106 pages, deals with equations of hyperbolic type. The first part deals with the Cauchy problem for non-analytic data, and introduces the idea of " reasonableness " (Korrektheit) of this problem; a problem is said to be reasonable if its solution depends continuously on the initial data. He discusses mainly the equation of wave motions in spaces of 3, 2 or 1 dimensions. The second part is concerned with the mixed initial and boundary value problem for the wave equation, and mainly with what he calls the general Fourier method (separation of variables).

The simplest representative of the class of elliptic equations, Laplace's equation, is discussed in Chapter III (68 pages), and most of the space is devoted to the theory of potential in the plane. The proof of the existence of a solution of Dirichlet's problem is that of Poincaré and Perron. Chapter IV of only 10 pages discusses very briefly some of the properties of the equation of heat, the typical equation of parabolic type.

Professor Duff's book is of roughly the same length as Professor Petrovsky's, but he covers much more ground. The argument has had to be condensed and much of the detail left to the reader. As a result, it is not an easy book to read, and I am inclined to think that he has overestimated the capabilities of the student for whom it is intended.

Professor Duff emphasises existence and uniqueness theorems, and introduces the reader in the first chapter to the Cauchy method of dominant power series for proving local existence and uniqueness theorems. The second and third chapters deal with first order equations and systems of such equations with one dependent and any number of independent variables. Most text books concern themselves with finding complete integrals, general integrals and singular integrals; but Duff rightly emphasises the geometrical side of the theory and introduces the method of characteristics at an early stage.

He goes on to linear equations of the second order of elliptic or normal hyperbolic type in his fourth chapter—equations of parabolic type are not considered. By using the ideas of Riemannian geometry, he writes the equation in tensor form

$$L[u] \equiv \Delta u + b^{\alpha} \frac{\partial u}{\partial x^{\alpha}} + cu = 0$$

where Δ is Beltrami's second differential parameter in the space of metric

$$ds^2 = a_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

and b^{α} is a contravariant vector, c a scalar. He then derives the Green's formula and introduces the operator M adjoint to L.

The chapter concludes with an account of the ideas of geodesic, geodesic distance and normal co-ordinates in Riemannian geometry; but this is, I think, too condensed for a reader previously unacquainted with the theory.

In Chapter V, the potential theory of the self-adjoint equation of elliptic type

$$\Delta u = qu$$

is discussed; here q is a positive function and Δ is Beltrami's second differential parameter for a space of positive definite metric. Hilbert's parametrix is introduced as an approximate fundamental solution, and the solution of boundary value problems is thereby reduced to the solution of integral equations of Fredholm's type. The Fredholm theory is given in Chapter VI and applied in Chapter VII. Uniqueness of the solutions is ensured if $q>0$. Relaxation of this condition leads to the consideration of Eigenvalue problems in Chapter VIII.

In Chapter IX, the theory of characteristic surfaces of the general linear equation of normal hyperbolic type is given in preparation for the solution of the problem of Cauchy for the wave equation in Chapter X, by Marcel Riesz's method.

Anyone who works systematically through Professor Duff's book will certainly gain a wide knowledge of the theory of partial differential equations. But unfortunately he gives very few references; there is admittedly a bibliography of 37 titles, but not all are relevant, and all but one are to treatises or monographs. There are still fewer references in Petrovsky's book and naturally many are to Russian sources.

The books are printed by the University Presses of Cambridge and Oxford respectively; that they are well produced goes without saying.

E. T. Copson

Relativity: the special theory. By J. L. Synge. Pp. 450. 76s. 1956. (North Holland Publishing Co. Amsterdam)

This is probably the most important book on relativity theory since the pioneering books of the early years of the subject. Professor Synge had, he says, planned a book on special and general relativity but he found so much to say on the special theory that no room was left for the general theory. Readers will share his confessed lack of regret at this outcome, but will also look forward to his soon giving them a second volume on the general theory.

The book is an entirely new development of the subject. Synge treats it as a theory in mathematical physics that " is to be judged primarily on the basis of the logical consistency and only secondarily on the basis of the truth of its physical predictions." Every worth-while branch of mathematical physics must probably reach this stage and it is certainly high time that such a treatment of special relativity should be available. There is no one better qualified than Synge to provide it. One of the book's most valuable features is that it includes a number of Synge's own notable contributions to the formulation of the subject from this point of view. These had not previously been as generally well-known as they deserve to be.

The book starts with a treatment of the space-time continuum that was evidently intended to lead on to the general theory as well, before this got crowded out. The geometry of the space-time of special relativity is then studied with great care, because Synge's aim is to present the whole subject primarily in terms of this geometry. He then gives a very general treatment of the Lorentz transformation and its immediate applications. The thoroughness of the discussion of these fundamentals can be judged from the fact that they occupy rather more than a third of the book.

The next third is devoted to relativistic mechanics. The mechanics of a particle is first developed. This includes collision-problems and several important particular cases, including ones involving photons, are worked out at length. Then the mechanics of a discrete system is studied. What makes this fundamentally different from Newtonian mechanics is the fact that the Newtonian law of action and reaction cannot be taken over into relativistic mechanics (except for particles in contact with each other). This is simply because the Newtonian law concerns action and *simultaneous* reaction, while in relativity simultaneity at distinct positions has no unique meaning. In order to supply a replacement in relativistic mechanics, Synge employs his concept of *internal impulses*. This enables him to develop a mechanics that is mathematically consistent with the rest of the theory. There is no necessity to enquire closely into the immediate physical significance of Synge's impulses themselves. They are reminiscent of some of the models used in nineteenth-century mathematical physics, as well as some of the very recent procedures used in quantum theory, in order to obtain results consistent with certain general requirements. Such procedures are rendered misleading only when someone forgets that they are originally no more than mathematical devices and attempts to regard them as physical hypotheses. In any case, Synge is careful to state that his treatment here is not part of the standard development of the subject. But this is probably because the problems he deals with by such means have not been adequately faced up to elsewhere. Synge further introduces the concept of *intrinsic angular momentum* of a particle. Mathematically, this is interesting because it renders the solution of collision-problems determinate, though not in general unique, and again Synge discusses a number of particular cases. Physically, it is interesting also because of the close correspondence with *spin* in quantum mechanics, though, of course, Synge does not imply that his results are identical with those of quantum theory.

All this then enables Synge to proceed through a statistical treatment to the formulation of the relativistic mechanics of a continuum and, in particular, to a highly satisfactory treatment of the energy-tensor.

The remainder of the book deals with relativistic electro-magnetism, first the treatment of the field in vacuo and then of fields and charges. Again the treatment of continuous distributions can be approached by a statistical treatment (*discrete method*) but here Synge also develops the *continuous method* directly, and then establishes the agreement between them.

Synge's discussion of the standard results of special relativity is almost certainly the most penetrating in existence. But his book includes also a number of excursions into topics that are less familiar in the context, as for instance the sections on spinors and spin-transformations, pp. 103–10. Synge is always fair to the reader in stating when he is being unorthodox, but the reader will find all these excursions to be stimulating and instructive.

At first sight the book is one for the specialist in relativity theory and without a doubt everyone wanting to specialize in the subject will have to study it. He ought, of course, to have some previous working knowledge of special relativity, sufficient at any rate to give him an idea of the scope and significance of the subject as well as to prepare him to appreciate the power of

Synge's methods. Incidentally, he ought to be warned that, as already mentioned, the discussion he will meet in Synge's first chapter is more general than what he requires for the rest of the work in this book.

As regards the applied mathematician or physicist who only wants either to get an idea of what relativity is about or to be able to use some of its simpler results, I still think that he would find some of the traditional treatments more useful. But these days anyone who more seriously concerns himself with the fundamentals of mathematical physics must learn to think in terms of the space-time of special relativity, whether or not he is interested in relativity theory for its own sake. For this purpose, he will find Synge's book to be of the utmost assistance. W. H. McCrea

An Introduction to Linear Algebra. By L. Mirsky. Pp. vii, 433. 35s. 1955. (The Clarendon Press, Oxford)

Great care has been taken by the author to write an introduction to linear algebra that is elementary, readable and self-contained. Though mainly intended for Honours students in Mathematics the book also caters for readers who are more interested in physics and technology. Although the author has confined himself to topics normally included in an undergraduate course, he has planned the work on a generous scale. He has not grudged the space for some didactic asides and for a certain amount of material which belongs more to a general mathematical background than to the particular subject at hand. Also the applications, notably those to geometry, are given more than a mere mention. For these reasons the volume has become larger than one might expect of an introductory text-book, but many a reader, and especially a beginner will prefer Dr. Mirsky's broad and congenial style to the concise and cold rigour which is so common in modern mathematical literature.

It is customary to distinguish roughly between two ways of presenting linear algebra: the formal or "old-fashioned" approach regards matrices as algebraical entities which obey certain rules of addition and multiplication. The properties of determinants are used to obtain an explicit expression for the inverse and to define the (determinantal) rank of a matrix. On the other hand, in an abstract or "modern" treatment the basic concepts are those of a vector space, linear transformation and bilinear form, whilst matrices appear only in a secondary role and determinants hardly at all. Dr. Mirsky aims at a compromise between these two points of view. He begins in the old style with an introductory chapter on determinants, to be followed by an exposition of matrix algebra, the formula for matrix multiplication being given without motivation at this stage. The rank of a matrix is first defined determinantally and is then proved to be equal to the row and column ranks. As the development proceeds, the abstract approach is introduced and the relation between the two methods is clearly brought out.

There is no doubt that the subject-matter of most undergraduate courses on linear algebra is amply covered. Yet, one cannot help regretting that in a work of this size the elementary divisor theory for integral matrices is relegated to three problems at the end of a chapter and that the Jordan canonical form is not done at all. The book on the whole reads well, but it is unfortunate that the initial chapter on determinants is perhaps the weakest. The discussion of the combinatorial preliminaries is heavy and the treatment, though elementary, is rather unattractive. With his expository skill, Dr. Mirsky would surely have succeeded in presenting the theory more elegantly by an alternative method, such as Weierstrass's approach which is sketched on pp. 189–92. Among the most valuable features of the book are the numerous examples and problems. The analyst will find the chapter on matrix analysis and the rich crop of inequalities particularly welcome. Altogether, this well produced volume will be appreciated by a large class of readers. W. Ledermann

Automation. By R. H. Macmillan. Pp. 100. 8s. 6d. 1956. (Cambridge University Press)

Macmillan traces the evolution of the automatic factory from the early controls like the centrifugal governor, the thermostat and the steam engine valve mechanism to the Doncaster ribbon machine for making glass envelopes for electric light bulbs and radio valves which transforms the sand and other new materials entering at one end into finished envelopes, packed and loaded directly into railway waggons at the other, the process being entirely automatic throughout. A chapter on automatic computers discusses the uses of computers but not the mathematical theory underlying them. The author sums up his position by saying that there are very definite limitations to what automation can achieve, that a complex feed back system may set up disastrous economic strains and that although automation may benefit labour—and usually does—this is not necessarily the case.

R. L. Goodstein

Problèmes d'Agrégation. By J. Dollon. Pp. 388. 1956. (Librairie Vuibert)

Agrégation is an examination set in France to those aspiring to be teachers in universities or grammar schools. The standard of difficulty is roughly that of the examination for the degree of M.A. or M.Sc. at London University.

This book consists of the geometrical papers set in the examination from 1919 to 1955, together with the solutions of most of them. Each paper consists of one long question (sometimes as long as $4\frac{1}{2}$ pages), divided into several parts, but all on one topic.

The point which one finds most striking about these papers is that they consist of very difficult problems on comparatively elementary work: the ground covered consists only of 2 and 3 dimensional Euclidean geometry, treated by Cartesian coordinates. One would have thought that candidates who are capable of solving problems of this difficulty could be using their energies more profitably by studying more modern work, such as projective geometry.

It is doubtful whether the book would be of much interest to British readers, except to those who enjoy solving problems for their own sake.

E. J. F. Primrose

THE MATHEMATICAL ASSOCIATION

Intending members of the Mathematical Association are requested to communicate with one of the Secretaries, Mr. F. W. Kellaway, Miss W. Cooke. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of *The Mathematical Gazette* and a copy of each new report as it is issued.

Change of Address should be notified to the Membership Secretary, Mr. M. A. Porter. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The Association Library is housed in the Library of the University of Leicester.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

CARDIFF BRANCH

Report for the Session 1956–1957

Officers : *President and Treasurer,* Mr. R. A. Jones ; *Secretary,* Mr. W. H. Williams.

There are 45 members and about 30 associate members.

Monday, 22nd October, 1956. The retiring President, Dr. R. Morris, of University College, Cardiff, gave an address, " Mathematics in Action : Aerodynamics ", in which she dealt with problems of lift and drag associated with the 2-dimensional aerofoil.

Monday, 19th November, 1956. Mr. N. D. Hayes, Head of the Cardiff Technical College Mathematics Department, gave an address on " Some Equations of Control Mechanisms and Economics " in which he showed how the same type of equations—" difference differential equations "—arise in such different fields as thermostatic controls, naval problems of stability, population problems in biology, and in theories of economics.

Monday, 21st January, 1957. Professor Gurney, of the Engineering Department, University College, Cardiff, gave an address on " Fatigue in Metals ", dealing with the ways in which fatigue is likely to produce fractures in such cases as the axle of a railway-truck and the wing of an aircraft, and indicating the ways in which these problems are being investigated.

Monday, 18th March, 1957. Dr. A. C. Bassett, the Examinations Officer of the Welsh Joint Education Committee, gave an address on " The G.C.E. Examinations, with special reference to Mathematics ", which provoked a very lively discussion.

W. H. Williams, *Hon. Sec.*

LEICESTER AND COUNTY BRANCH

Report for the Session 1956–1957

Nine meetings have been held by the branch this session, all at the University. They were as follows :

" Language, Logic and Mathematics " by Professor J. W. Tibble of Leicester University.

" Mathematics in Comprehensive Schools " by Miss Y. B. Guiseppi of the Dick Sheppard School, London.

" The Teaching of Mathematics in Primary Schools "—a discussion on the recently published Association report, opened by Mrs. E. M. Williams of Whitelands College, Putney, and Dr. S. Semple of the Leicester Institute of Education. This meeting was well attended by Primary School teachers from the City and County, and discussion, after the opening speakers' remarks, was lively and prolonged.

" The Propositional Calculus and Thinking Machines " by Mr. A. Rose of Nottingham University.

" The Shortage of Mathematicians "—a discussion opened by Mr. A. P. Rollett, H.M.I.

" Applied Mathematics in Schools and Universities " by Mr. R. Buckley of Leicester University—a talk which provoked much discussion from Grammar School teachers present.

The Annual General Meeting, followed by members' papers. Mr. W. A. Christie spoke on " The Mathematics Room ", Mr. J. T. Dudley on " A Plea for the Parabola ", and a paper by Mr. E. C. Witcombe on " When is a linear equation not a linear equation?" was read, in the author's absence, by the Secretary.

Two meetings were arranged especially for members of sixth forms. Both were well attended and we are hoping to arrange two such meetings in future sessions. The first was the annual quiz between City and County schools—a popular event, enjoyed more, perhaps, by the audience than by members of the teams. At the second, Mr. K. F. Solloway, speaking on the subject " See for Yourself ", demonstrated his unique collection of models illustrating mathematical facts and principles.

The Modern Schools discussion group mentioned in our last report has continued to meet and a Primary Schools discussion group has recently been formed to consider problems arising from the teaching of Mathematics in Infant and Junior schools.

Branch membership has again increased. There are now thirty full members and approximately fifty-five associates.

Mr. W. Flemming, of the Education department at Leicester University, who has been President of the branch since 1951 and who has done so much to foster its growth and stimulate its activities, resigned his office at our business meeting. He has been succeeded by Mr. J. W. Hesselgreaves, of Wyggeston Boys' School, Leicester. Other office-holders for the 1957–1958 session are : *Vice-Presidents,* Professor R. L. Goodstein, Mr. W. Flemming and Mr. T. R. Goddard ; *Secretary,* Mr. W. E. Remington ; *Treasurer,* Mr. L. G. Clarke.

W. E. Remington, *Hon. Sec.*

THE LIVERPOOL MATHEMATICAL SOCIETY

Report for the Session 1956–1957

Officers : *President,* Mr. A. Young ; *Vice-President,* Rev. R. A. S. Martineau ; *Treasurer,* Mr. E. J. Watson ; *Secretary,* Dr. G. R. Baldock ; *Committee :* Mr. R. Bromley, Dr. A. Fletcher, Miss L. Harvey, Mr. W. O. Macdonald, Mr. D. Temple Roberts, Miss D. B. Walker, Dr. T. J. Willmore.

Report of Meetings

22nd October, 1956. Several members of the staff of Burtonwood American School were invited to a discussion on " Secondary School Mathematics in the U.S.A. and Britain ". The Principal of the school, Dr. Christiansen, provided an introduction in which he described the curricula and the arrangements of courses in American schools. An extensive discussion followed.

12th November, 1956. Dr. Scorer, of Imperial College, London, spoke on "Mountain Waves in the Atmosphere ". He explained how a smooth hump on horizontal ground could produce a disturbance in the flow of air extending to a great height. The disturbance is followed by a series of waves which may become stabilized and be evident in a series of almost stationary bands of cloud through which the air stream passes. This phenomenon was shown speeded up by means of some cinema films, and other illustrations were provided by lantern slides.

3rd December, 1956. A lecture entitled " On Tangents to Curves " was given by Dr. Flett. The commonly used definition of a tangent to a curve in books on calculus possesses many unsatisfactory features which could cause confusion. Having examined several other possible definitions, Dr. Flett concluded by recommending that the simplest of them be adopted in the teaching of elementary calculus.

21st January, 1957. Dr. P. J. Hilton of Manchester addressed the Society on " George Bidder ". Among calculating prodigies, Bidder is of exceptional

interest because he retained his powers not only during his period of formal education but throughout his life. He was thus able to record some of his methods, and Dr. Hilton gave some illustrations of them. He concluded by suggesting that most of the calculators succeeded only by excluding from their minds everything except arithmetical facts. It was thus not surprising that either their powers declined when they began during their education to employ their brains more conventionally, or they retained the gift of calculation at the expense of appearing otherwise unintelligent.

11th March, 1957. Professor Kopal addressed the Society on "Keeping the Time". He surveyed the history of time measurement from antiquity to the present day. He explained how major advances in accuracy have been achieved firstly by abolishing mechanical linkage between the motor and the regulator, and secondly by reducing the size of the vibrating element to make its action independent of external disturbances. He concluded by discussing two applications; the possibility of predicting earthquakes, and the test of the time dilatation hypothesis in the theory of general relativity.

6th May, 1957. After the election of the Council, the Treasurer's interim report was presented to the Annual General Meeting and adopted. Mr. Young gave his Presidential Address which he entitled "On Doing Arithmetic". He sketched the development of number systems and arithmetical methods from the earliest times up to the present day. He mentioned several aids to calculation, ranging from the tally stick to the modern automatic computor, and showed how the method employed in a calculation must be suited to the type of machine and to the experience of the operator.

Council for the Session 1957–1958

At the Annual General Meeting the following members were elected to serve on the Council during the next session:

President, Rev. R. A. S. Martineau; *Vice-President*, Dr. T. J. Willmore; *Treasurer*, Dr. W. F. Newns; *Secretary*, Dr. G. R. Baldock; *Committee*, Mr. R. Bromley, Dr. A. Fletcher, Miss L. Harvey, Mr. J. Kershaw, Mr. W. O. Macdonald, Miss D. B. Walker, Mr. A. Young (*ex officio*).

MANCHESTER BRANCH

Report for 1955–1956

27th September, 1955. The Annual General Meeting was followed by a paper on "The Integration of Ordinary Mathematics, Syllabus II" given by Mr. P. F. Burns.

15th November, 1955. Mr. A. P. Rollett gave a talk on "Starting Geometry".

24th January, 1956. Miss E. M. Holman and Mr. J. C. Blakey, two of our own members, spoke on "The Mathematics Library".

20th March, 1956. Mr. R. W. Morris spoke on "The Place of Trigonometry in the School Course".

15th May, 1956. We held our joint meeting with the Mathematical Society, at the University, when Dr. E. A. Maxwell gave a talk entitled "On Going Wrong".

17th July, 1956. We held our annual discussion of the N.U.J.M.B. papers in Mathematics.

The Officers elected for 1956–1957 are: *President*, Miss J. M. Cawley; *Treasurer*, Miss E. Berry; *Secretary for* 1958 *Conference*, Mr. D. R. Wilcox.

M. Owen, *Branch Secretary.*

MANCHESTER AND DISTRICT BRANCH

Report for the Year 1956–1957

At the Annual General Meeting in September, Miss J. M. Cawley was elected President. Other officers were: *Hon. Treasurer,* Miss E. Berry; *Hon. Secretary,* Miss M. Owen; *Committee,* Misses D. Bannister, F. M. Holgate, E. M. Holman, Messrs. B. B. Cooper, R. H. Cripwell, J. H. Dixon, A. I. Gregory and S. T. Bates.

Miss E. M. Holman is the Secretary for the A.G.M. in Manchester, April 9th–12th, 1958.

Six meetings were held, as follows:

25th September, 1956. Annual General Meeting, followed by a paper entitled "Mathematics—A Simple Presentation", given by Miss E. E. Biggs.

20th November, 1956. Dr. B. H. Neumann, of Manchester University, spoke on "What the University expects from entrants in Pure Mathematics".

22nd January, 1957. Dr. J. Gani, of Manchester and Australia, gave "An Introduction to Statistics".

19th March, 1957. Dr. T. A. S. Jackson, of Liverpool University, spoke on "Friction".

28th May, 1957. We held our first joint meeting with the Manchester Mathematical Society, at the University, and were privileged to hear Dr. Joseph Needham, F.R.S., speak on "Mathematics and Science in Chinese Civilisation".

16th July, 1957. The 1957 N.U.J.M.B. papers in Mathematics were discussed.

The average attendance at the Branch meetings was 45, an increase of 6 on the previous Session.

M. Owen, *Hon. Sec.*

NORTH STAFFORDSHIRE BRANCH

Report for the Session 1956–1957

The Branch has had a successful year. Most of the meetings have been well attended and we have been pleased to welcome a good number of sixth form visitors.

Two of the meetings took the form of discussions on:

1. "Careers open to Mathematicians". A variety of ideas was given and the Branch expressed a strong desire that the Association should draw up a list of suitable careers.

2. Papers set in the summer G.C.E. examinations by the N.U.J.M.B. and the O. and C. Joint Boards.

At the other meetings the speakers and subjects were:

Mr. G. A. V. Leaf, "The Applications of Mathematics to the Hosiery Industry".

Dr. A. Illiffe, "Intelligence Tests".

Professor Sneddon, "Robert Simpson".

For the Summer meeting some members paid a very interesting visit to the Nelson Research Laboratory at Stafford.

M. V. Greenway

THE YORKSHIRE BRANCH

Report for the Session 1956–1957

The Annual General Meeting was held on:

20th October, 1956. Miss S. A. Wallbank, Lecturer in Education, University of Leeds, the retiring President, addressed the Branch on "The Old and the New in Mathematics Teaching".

Other meetings were held as follows:

24th November, 1956. Dr. E. A. Maxwell, Fellow of Queens' College, Cambridge, on "The Teaching of Higher Geometry in Schools".

9th March, 1957. Dr. T. A. S. Jackson, Lecturer in Applied Mathematics, University of Liverpool, on "Friction".

18th May, 1957. Mr. J. C. Boustead, Vice-Principal, British Railways Staff Training College, Derby, on "Railway Timetables".

In the evenings of:

20th, 27th February and 6th, 13th March, 1957. Mr. G. S. James, Lecturer in Statistics, University of Leeds, gave four lectures on "Finite Arithmetic".

All the above meetings were held in Leeds.

Officers for the session have been: *President*, Mr. W. Taylor; *Vice-Presidents*, Miss E. Rimmer, Prof. T. G. Cowling; *Secretary*, Mr. L. K. Turner; *Treasurer*, Miss B. Black.

The present membership is about 70 and the average attendance at the four Saturday meetings was 40.

L. K. Turner, *Secretary*

NOTTINGHAM BRANCH

Report for the Session 1955–1956

The Annual General Meeting of the Branch was held on Saturday, 3rd December, 1955, when the speaker was Dr. Snowdon on the subject "Field Analogies". Dr. Snowdon explained that when two problems had a common underlying Mathematical framework, although the physical quantities differed the differential equations obtained were the same, and consequently it is possible to solve scientific problems by using analogies. He showed first how the same contour diagram derived from a solution of Laplace's Equation represented many different problems, and went on to describe the use of analogies when analytical solutions cannot be found. Experiments of analogous phenomena are often easily conducted, especially using current electricity, elasticity and thin films of viscous fluids, and Dr. Snowdon gave several illustrations which showed this method to be a very powerful practical one.

At the Spring meeting on Saturday, 17th March, the Branch was addressed by Professor Broadbent who had recently returned from an international conference on Mathematical Education held in Bombay. He reported on the work of the Conference which was mainly concerned with education in S.E. Asia, and painted a vivid picture of the difficulties facing these countries in dealing with an illiterate population with very few teachers. Addresses were given by delegates from many countries, but the most important work was done in the working committees which were set up to consider the aim and content of Mathematical teaching, methods to be adopted and training of teachers at the School, University and Post Graduate stages. Professor Broadbent discussed the findings of these groups, which were to be brought

to the attention of the Governments of S.E. Asian countries, and he was certain that substantial good would result from the Conference.

Dr. Davy was the speaker at the Summer Meeting, held on Thursday, 21st June. He spoke on " Some Mathematical Physicists of the Napoleonic Era ", and pointed out that the French Revolution was the means of giving many the chance to rise which would otherwise have been denied them. This was especially true in military affairs and in Science where those who showed ability rose very rapidly, and accounted for the large numbers of prominent scientists in France at a time when there were very few elsewhere in Europe. Dr. Davy picked out Fourier, Arago, and Count Rumford, discussing their work and their personalities, with amusing anecdotes concerning their private lives. He illustrated his talk with slides and with demonstration experiments.

The Officers of the Branch were: *President*, Dr. G. Power; *Vice-President*, Mr. K. R. Imeson; *Hon-Sec.*, Mr. F. E. Chettle; *Hon Treas.*, Mr. C. R. Swaby; *Committee*, Misses Onions, Hill, Dr. Jackson, Messrs. Kirshner, Slack, Melluish.

QUEENSLAND MATHEMATICAL ASSOCIATION

Report for the Session 1956–1957

The Association records with regret the death in February 1957, of Mr. R. A. Kerr, a foundation member and one of our first Vice-Presidents. Of late years Mr. Kerr had been an honorary member of the Association.

The Annual Meeting was held at the University, George Street, on 20th March, 1956. The Annual Report and the Statement of Receipts and Expenditure were presented to the meeting and both were adopted.

This was followed by the election of officers for the coming year, and then Mr. H. K. Powell, M.A., B.Sc., read a paper entitled " Isogonal Conjugates ".

Three general meetings were held during the year, all at the University, George Street. At the first, on 22nd May, Mr. E. J. Burr read a paper entitled " Ratings of Skill in Two-person Games ". The second was held on 31st July, at which Mr. P. B. McGovern, M.A., B.Sc., read a paper entitled " Solution of Linear Equations and Evaluation of Determinants: Exact Methods ". At the third, held on 30th October, Professor C. S. Davis, M.Sc., Ph.D., read a paper on " Approximating Polynomials ".

The Statement of Receipts and Expenditure for the year discloses a credit balance of £11 14s. 3d. Most of the expenses are incurred in postal charges, incurred in connection with the circulation of the *Mathematical Gazette* among members.

The number of members is at present 43, which includes 2 Life Members of the Mathematical Association and 15 Ordinary Members of that Association. 6 new members have been admitted during the year, and there have been 3 resignations, all due to the transfer from Brisbane of the members concerned.

The *Mathematical Gazette* arrives regularly and is circulated as quickly as circumstances allow, among the Associate Members. During the year, copies of the *Gazette* for 1951, 1952, 1953, and 1954 have been placed in the Main University Library.

The attendance at meetings has been satisfactorily maintained, and the interesting papers read at the various meetings have led to good discussions. The Association is very grateful to those members who provide papers and initiate discussions.

Myra A. Popple, *Hon. Secretary.*

MATHEMATICAL ASSOCIATION OF VICTORIA

Report for the Session 1955–1956

This was the Jubilee Year of the Association, and special functions were held to celebrate the anniversary. Professor T. M. Cherry was President for 1956.

The special Jubilee meeting in April was notable for the conferring of Honorary Life Membership on Professor T. M. Cherry and Mr. F. J. D. Syer, in recognition of their long and distinguished service to the Association. Miss A. Laing, who was at the first meeting in June 1906, also attended this Jubilee meeting, to the great pleasure of all present.

The special activities organized by the Jubilee Committee which was appointed in 1954 for these purposes were :

A residential Refresher School for teachers of Mathematics which was held during a week-end in May at Caulfield Grammar School. 62 members attended. The guest speaker was Dr. A. R. Hogg, Commonwealth Acting-Astronomer.

A Mathematics Exhibition at the University, lasting for a week in August which drew about 3,000 visitors.

Compilation and editing of the three numbers of the *Australian Mathematics Teacher* for 1956, published by the New South Wales Branch of the Association. This work is not yet completed.

Compilation of a symposium on the history of the teaching of Mathematics during the last 50 to 80 years. This work is also not yet completed.

The Jubilee Committee proposes to submit a report on these activities at some subsequent date. Meantime this Jubilee Committee and all those who assisted it, especially the exhibition workers, merit the congratulations and thanks of the Association.

During the year the Committee met three times, and general meetings were held on six occasions (including the Annual Meeting). In addition, a special meeting was held on the occasion of the visit to Melbourne of Professor A. W. Tucker, of Princeton University, U.S.A.

The programme for the year was :

March	" The New Syllabus in Geometry for Intermediate Mathematics A and Mathematics B, with the effect on Leaving Mathematics II ". Professor E. R. Love led the discussion ; other speakers were Messrs. A. E. Schruhm, R. Rowlands and F. R. Manley.
April	Special Jubilee Meeting : Professor T. M. Cherry and Mr. F. J. D. Syer spoke on the history of the Association.
May	Refresher School held at Caulfield Grammar School, organized by the Jubilee Committee.
June	" Mathematics for the Electrical Engineer ". Professor C. E. Moorhouse.
July	" Teaching Mathematics in U.S.A.". Mr. H. J. Russell.
August	Mathematical Exhibition—organized by the Jubilee Committee. Also a special meeting : " New Patterns in Mathematical Education ". Professor A. W. Tucker.
September	Presidential Address : " The Mathematics of the Motor Car ". Professor T. M. Cherry.
October	Annual Meeting.

The total membership is 196, an increase of 16 since last year. The financial membership is 134, made up of 80 full and institutional members, and 54 associate members.

Average attendance was 76, compared with 59 in 1955. Three afternoon meetings were held, and again were popular, with 190 the greatest attendance.

The sale of Matriculation examination solutions was repeated, with Messrs. Ryan and Halstead being responsible for duplicating, and Mr. Russell undertaking the work of postage. The Association again expresses thanks to the anonymous people who supplied the solutions.

Mrs. Hutton again occupied the position of Librarian ; members are glad to know that she has recovered from her illness.

Mr. R. L. Harrison and Mr. H. J. Russell have shared the duties of Recorder this year.

The magazines of the Associations of teachers of mathematics in U.S.A., France and Germany have been received during the year.

The death occurred in June of Mr. D. K. Picken, former Master of Ormond College, and for many years one of the leading members or the Association. Mr. Picken occupied the position of President for six years, 1938-1944 ; he addressed the Association on eleven occasions, and was a very enthusiastic member for many years. An appreciation of his work for the Association is appended to this report.

The Secretary again wishes to express his pleasure at the willingness with which members have co-operated during 1956. While this spirit of cheerful service is evident, the activities of the Association are certain to be successful.

F. Roy Manley, *Hon. Secretary.*

OBITUARY

David Kennedy Picken, who died on 17th June, 1956, was a member of the Mathematical Association for 51 years and a regular contributor to the *Gazette*. He was a Scotsman, trained in Glasgow and Cambridge, where in 1902 he was sixth wrangler. After graduation he was successively Lecturer in Mathematics at Glasgow University, and Professor at Victoria College, New Zealand. In 1915 he was called to be Master of Ormond College, residential and ancillary teaching institution affiliated with the University of Melbourne. In that position he had wide administrative duties, and the moral responsibilities attaching to the care of more than 100 undergraduates and a leading lay position in the Presbyterian Church. In a sense therefore he ceased to be a professional mathematician. But the things that interested Picken in mathematics were the fundamental principles—the " elements ": their nature, their proper expression, and the proper exposition of the developments stemming from them. On these things he had deep and even passionate convictions, and it seems that these were to some extent linked with his faith regarding the fundamentals of religion and conduct (the comparison of the " infinities " of mathematics and faith would provide such a link); it is understandable therefore that in mathematical thought and teaching he remained active. In the teaching world he had for many years a responsible position on the Board which prescribes " school certificate " syllabuses and recommends courses of study, and he was an influential member of the bodies which deal similarly with studies within the University of Melbourne ; but his more intimate care was for the mathematical well-being of the students in his College, and on many of these he had a decisive influence.

To illustrate these generalities : We all know that multiplication is one of the fundamental operations, and that in omitting the sign of multiplication (.) from expressions such as $a\,.\,b$ of $2\,.ft$ we are departing from our practice concerning the sign + of addition. But for Picken, to omit the multiplication sign was " wrong " notation. This example may be misleading by suggesting that Picken's dominant concern was with what most people would call trivialities, but it does give a fair indication of the flavour of his writing. Examples may be found in the *Gazettes* over many years ; for one of the last see Vol. XXX

(1946), p. 200. His published books are *The Theory of Elementary Trigonometry* (Whitcombe and Tombs, 1910), *The Number System of Arithmetic and Algebra* (Melbourne Univ. Press, 1923), and (with Miss Winifred Waddell) *A First Trigonometry* (Melbourne, 1919).

In the field of elementary geometry Picken made a remarkable discovery: the unifying concept of the " complete angle ". It is explained, and its utility shown, in the first chapter of Forder's *Higher Course Geometry*, being there called " the cross ". Picken's original paper is in *Proc. London Math. Soc.* [2], 23 (1925), 45-55.

Picken was of course a prominent member of the Mathematical Association of Victoria—which incidentally has this year (1956) celebrated its Jubilee ; from 1938 to 1944 he was its President. His addresses to the Association, on topics such as The Number System, Ratio and Proportion, and The Quantities of Physics are substantially embodied in his contributions to the *Gazette*.

T. M. C.

MIDLAND JUNIOR MATHEMATICAL SOCIETY

Report for the Quinquennium 1952-1957

In October 1952 a group of enthusiastic mathematicians met at King Edward's School, Birmingham, under the chairmanship of Mr. M. A. Porter, to discuss the launching of the above Society. The original aim was to hold meetings at Birmingham University and arrange outings to places of interest for boys and girls in Midland Schools who study advanced mathematics.

Three main meetings were held in the first year. The inaugural speakes in October 1952 was Mr. W. Hope-Jones who lectured on " Some Fun with Probability ". In November Mr. G. A. Montgomerie addressed the Society on " Calculating Machines " and brought with him some demonstration models. Four members read papers in March 1953. With the kind co-operation of Prof. T. A. A. Broadbent, an outing to the Royal Naval College at Greenwhich took place in the Summer.

During the Session 1953–1954 three meetings were held. The visiting speakers were Mr. A. P. Rollett who lectured on " The Series of Fibonacci " (November 1953) and Dr. E. A. Maxwell on " Geometry " (February 1954). Some members' papers were delivered at the May meeting. A social evening was held successfully in January. There were visits to the Aston collection of models in March and to the Mathematical Laboratory at Cambridge in July, when Dr. E. A. Maxwell was our host, entertaining us to lunch at Queens' College.

Over the period 1954–1955, four meetings were held. There was a social evening in January 1955 and an outing to the Royal Aircraft Establishment, Farnborough, in the Summer. Visiting speakers at the meetings were Mr. E. V. Smith on " Statistics " (November 1954), Dr. R. W. H. Small on " Crystallography " and Prof. C. A. Rogers on " Types of Mathematics " (May 1955).

By the end of 1955 the Society had a steady regular attendance at meetings of some 40-50 members who came from the principal Public and Grammar Schools in Birmingham and the Midlands, and from as far afield as Warwick, Bromsgrove, Dudley, Kidderminster and Sutton Coldfield. As the funds were stabilised for the first time it was decided at this stage to hold six main meetings per session : two each term.

Visiting speakers and their subjects were as follows :

Prof. T. A. A. Broadbent—" George Boole and his Algebra " (October 1955),

Mr. W. O. Storer—" Mathematics in Music " (November 1955), Mr. Ball—" Visual Illusion " (February 1956), Prof. I. N. Sneddon—" Mathematics in Biology " (March), Mr. W. R. Solloway—" Mathematics as an Art " (May), and Mr. G. A. Montgomerie—" Calculating Machines " (June). A highly successful social evening attended by about 80 members took place in January, and in July there were outings to Harwell and R.A.E., Farnborough.

The first meeting of the present Session was held in October 1956, when Dr. J. A. H. Waterhouse spoke to a packed audience on " Statistics ". In November Mr. Reynolds delivered a lecture on " The Determination of Astronomical Distances ". A party of 30 members visited R.A.E., Farnborough. The Annual Social Evening was held in January at King Edward's High School for Girls, Edgbaston. There was a programme of games, mathematical competitions and dancing. The event was well attended and thoroughly enjoyed.

In February Mr. W. Curr lectured to the Society on " Intelligence Tests ". In March Dr. A. H. Wallace spoke on " Axiom Systems ". This term there has been a talk on " Probability " by Mr. W. Hope-Jones (May) and on June 1st Mr. W. O. Storer will address the Society on " Relativity ". Outings to the Mathematical and Cavendish Laboratories at Cambridge and to R.A.E., Farnborough, have been arranged.

From the original nucleus of 25 in 1952, the Society has expanded to a present record membership of 164, which consists of equal proportions of boys and girls. The annual subscription is 1s. 6d. Our warmest and sincerest thanks are hereby accorded to all who have so kindly sacrificed Saturday afternoons to come to Birmingham to address the Society. Without their indispensible help in delivering such stimulating lectures of the highest quality, the Society could not conceivably have become so flourishing a body.

The Officers of the present 1956–1957 Session are: *Chairman*, Mr. F. Dean; *Adult Committee Members*, Miss L. E. Hardcastle, Miss F. M. L. Tebbutt, Mr. W. O. Storer; *Secretary and Treasurer*, M. J. Short.

M. J. Short, *Hon. Secretary.*

WANTED

H. G. Forder: *School Geometry.*

H. G. Forder: *Higher Course Geometry.*

T. P. Nunn: *Exercises in Algebra*, Part II.

Please write to A. G. Sillitto, Jordanhill Training College, Glasgow, W. 3.

A member requires urgently to borrow or purchase answers to the Cambridge School Certificate Papers 1933–42 for Algebra, Geometry and Arithmetic, and to the corresponding papers of the Northern Universities J.M.B. School Certificate 1946–51. Replies to J. C. Smith, 35 Fulneck, Near Pudsey, Leeds (Pudsey 5103).

FOR SALE

A collection of about 50 mathematical text books (by Salmon, Frost, Routh, Hobson and others) and a few books on Astronomy and Physics. H. Price, 85 Marlborough Mansions, London, N.W. 6.

Mathematical Gazette, Nos. 322 to 335 and Nos. 308, 310. T. C. Grice, Ipswich School, Henley Road, Ipswich.

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